EXPERIMENTAL PROBLEM 1

DETERMINATION OF THE WAVELENGTH OF A DIODE LASER

MATERIAL

In addition to items 1), 2) and 3), you should use:

4) A lens mounted on a square post (LABEL C).
5) A razor blade in a slide holder to be placed in acrylic support, (LABEL D1) and mounted on sliding rail (LABEL D2). Use the screwdriver to tighten the support if necessary. See photograph for mounting instructions.
6) An observation screen with a caliper scale (1/20 mm) (LABEL E).
7) A magnifying glass (LABEL F).
8) 30 cm ruler (LABEL G).
9) Caliper (LABEL H).
10) Measuring tape (LABEL I).
11) Calculator.
12) White index cards, masking tape, stickers, scissors, triangle squares set.
13) Pencils, paper, graph paper.

Razor blade in a slide holder to be placed in acrylic support (LABEL D1) and mounted on sliding rail (LABEL D2).

EXPERIMENT DESCRIPTION

You are asked to determine a diode laser wavelength. The particular feature of this measurement is that no exact micrometer scales (such as prefabricated diffraction gratings) are used. The smallest lengths measured are in the millimetric range. The wavelength is determined using light diffraction on a sharp edge of a razor blade.

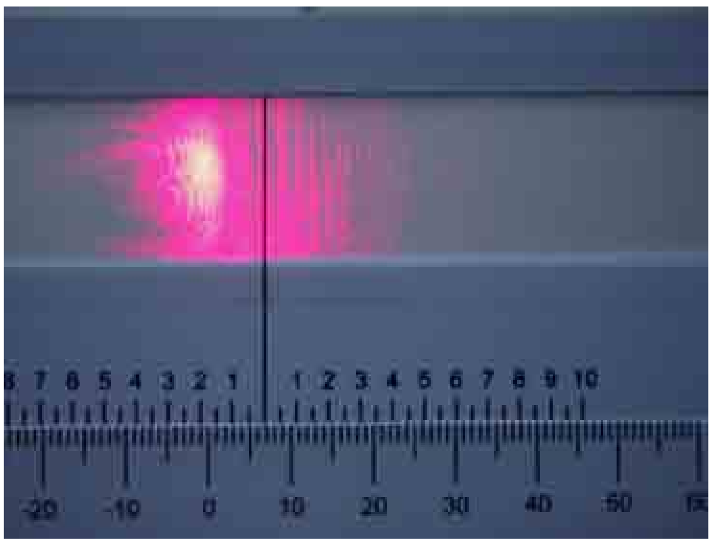


Figure 1.1 Typical interference fringe pattern.

Once the laser beam (A) is reflected on the mirror (B), it must be made to pass through a lens (C), which has a focal length of *a few centimeters*. It can now be assumed that the focus is a light point source from which a spherical wave is emitted. After the lens, and along its path, the laser beam hits a sharp razor blade edge as an obstacle. This can be considered to be a light source from which a cylindrical wave is emitted. These two waves interfere with each other, in the forward direction, creating a diffractive pattern that can be observed on a screen. See Figure 1.1 with a photograph of a typical pattern.

There are two important cases, see Figures 1.2 and 1.3.

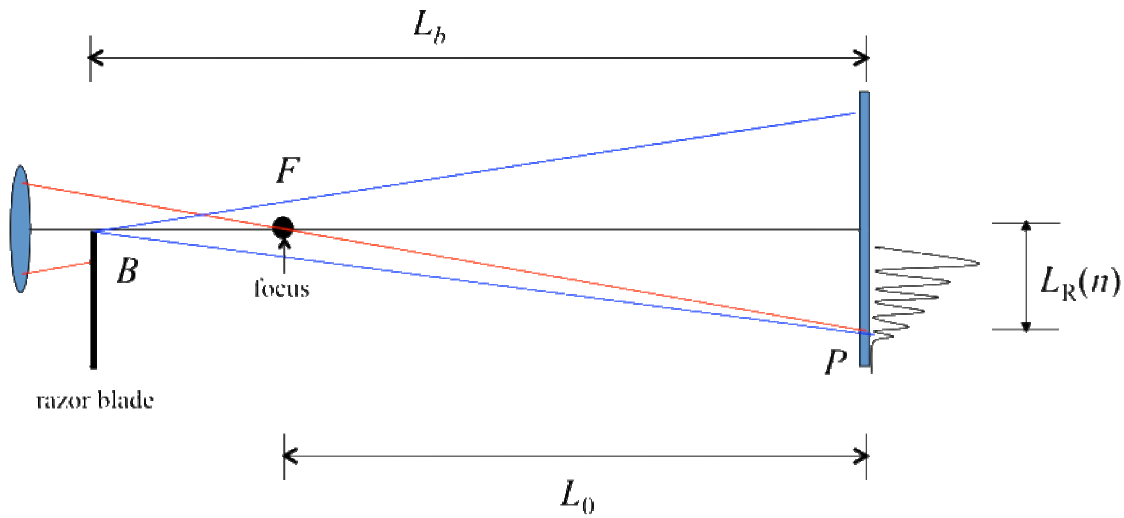


Figure 1.2. Case (I). The razor blade is *before* the focus of the lens. Figure is not at scale. B in this diagram is the edge of the blade and F is the focal point.

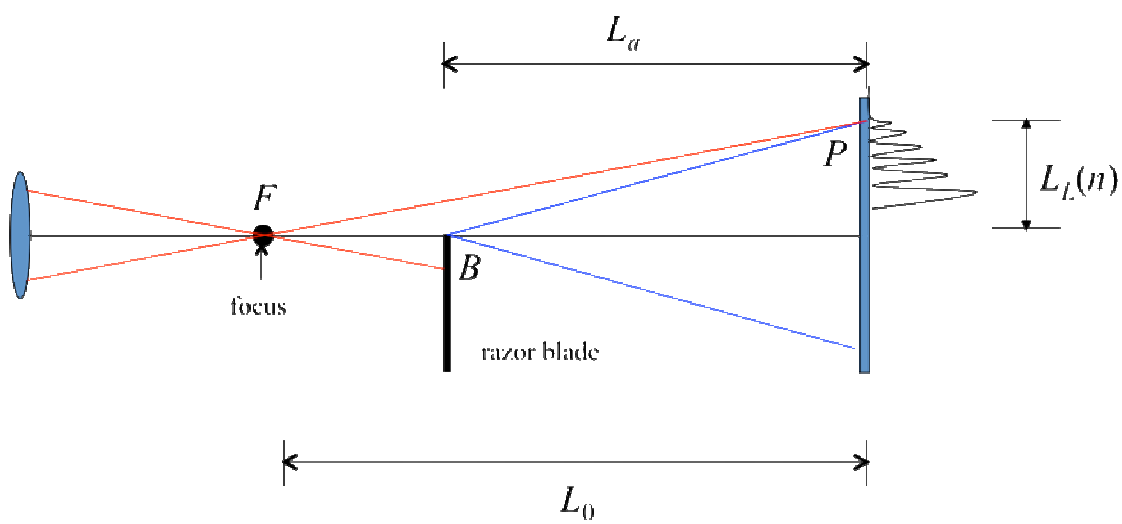


Figure 1.3. Case (II). The razor blade is *after* the focus of the lens. Figure is not at scale. B in this diagram is the edge of the blade and F is the focal point.

EXPERIMENTAL SETUP

Task 1.1 Experimental setup (1.0 points). Design an experimental setup to obtain the above described interference patterns. The distance L_0 from the focus to the screen should be much larger than the focal length.

- Make a sketch of your experimental setup in the drawing of the optical table provided. Do this by writing the LABELS of the different devices on the drawing of the optical table. You can make additional simple drawings to help clarify your design.
- You may align the laser beam by using one of the white index cards to follow the path.
- Make a sketch of the laser beam path on the drawing of the optical table and write down the height h of the beam as measured from the optical table.

WARNING: Ignore the larger circular pattern that may appear. This is an effect due to the laser diode itself.

Spend some time familiarizing yourself with the setup. You should be able to see of the order of 10 or more vertical linear fringes on the screen. The readings are made using the positions of the **dark** fringes. You may use the magnifying glass to see more clearly the position of the fringes. **The best way to observe the fringes is to look at the back side of the illuminated screen (E).** Thus, the scale of the screen should face out of the optical table. If the alignment of the optical devices is correct, you should see both patterns (of Cases I and II) by simply sliding the blade (D1) through the rail (D2).

THEORETICAL CONSIDERATIONS

Refer to Figure 1.2 and 1.3 above. There are five basic lengths:

L_0 : distance from the focus to the screen.
L_b : distance from the razor blade to the screen, Case I.
L_a : distance from the razor blade to the screen, Case II.
$L_R(n)$: position of the n-th **dark** fringe for Case I.
$L_L(n)$: position of the n-th **dark** fringe for Case II.

The first dark fringe, for both Cases I and II, is the widest one and corresponds to $n = 0$.

Your experimental setup must be such that $L_R(n) << L_0, L_b$ for Case I and $L_L(n) << L_0, L_a$ for Case II.

The phenomenon of wave interference is due to the difference in optical paths of a wave starting at the same point. Depending on their phase difference, the waves may cancel each other (destructive interference) giving rise to dark fringes; or the waves may add (constructive interference) yielding bright fringes.

A detailed analysis of the interference of these waves gives rise to the following condition to obtain a **dark** fringe, for Case I:

$$\Delta_{\rm I}(n)=\left(n+\frac{5}{8}\right)\lambda \qquad \text{with} \qquad n=0,\,1,\,2,\,\ldots \qquad (1.1)$$

and for Case II:

$$\Delta_{\rm II}(n)=\left(n+\frac{7}{8}\right)\lambda \qquad \text{with} \qquad n=0,\,1,\,2,\,\ldots \qquad (1.2)$$

where λ is the wavelength of the laser beam, and $\Delta_{\rm I}$ and $\Delta_{\rm II}$ are the optical path differences for each case.

The difference in optical paths for Case I is,

$$\Delta_{\rm I}(n)=(BF+FP)-BP \qquad \text{for each} \qquad n=0,\,1,\,2,\,\ldots \qquad (1.3)$$

while for Case II is,

$$\Delta_{\rm II}(n)=(FB+BP)-FP \qquad \text{for each} \qquad n=0,\,1,\,2,\,\ldots \qquad (1.4)$$

Task 1.2 Expressions for optical paths differences (0.5 points). Assuming $L_R(n) << L_0, L_b$ for Case I and $L_L(n) << L_0, L_a$ for Case II in equations (1.3) and (1.4) (make sure your setup satisfies these conditions), find approximated expressions for $\Delta_{\rm I}(n)$ and $\Delta_{\rm II}(n)$ in terms of L_0, L_b, L_a, $L_R(n)$ and $L_L(n)$. You may find useful the approximation $(1+x)^r \approx 1+rx$ if $x << 1$.

The experimental difficulty with the above equations is that L_0, $L_R(n)$ and $L_L(n)$ cannot be accurately measured. The first one because it is not easy to find the position of the focus of the lens, and the two last ones because the origin from which they are defined may be very hard to find due to misalignments of your optical devices.

To solve the difficulties with $L_R(n)$ and $L_L(n)$, first choose the zero (0) of the scale of the screen (LABEL E) as the origin for all your measurements of the fringes. Let l_{0R} and l_{0L} be the (unknown) positions from which $L_R(n)$ and $L_L(n)$ are defined. Let $l_R(n)$ and $l_L(n)$ be the positions of the fringes as measured from the origin (0) you chose. Therefore

$$L_R(n)=l_R(n)-l_{0R} \qquad \text{and} \qquad L_L(n)=l_L(n)-l_{0L} \qquad (1.5)$$

PERFORMING THE EXPERIMENT. DATA ANALYSIS.

Task 1.3 Measuring the dark fringe positions and locations of the blade (3.25 points).

- For both Case I and Case II, measure the positions of the dark fringes $l_R(n)$ and $l_L(n)$ as a function of the number fringe *n*. Write down your measurements in Table I; you should report no less than 8 measurements for each case.
- Report also the positions of the blade L_b and L_a, and indicate with its LABEL the intrument you used.
- **IMPORTANT SUGGESTION:** For purposes of both simplification of analysis and better accuracy, measure *directly* the distance $d = L_b - L_a$ with a better accuracy than that of L_b and L_a; that is, do not calculate it from the measurements of L_b and L_a. Indicate with its LABEL the instrument you used.

Make sure that you include the uncertainty of your measurements.

Task 1.4 Data analysis. (3.25 points). With all the previous information you should be able to find out the values of l_{0R} and l_{0L}, and, of course, of the wavelength λ.

- Devise a procedure to obtain those values. Write down the expressions and/or equations needed.
- Include the analysis of the errors. You may use Table I or you can use another one to report your findings; make sure that you label clearly the contents of the columns of your tables.
- Plot the variables analyzed. Use the graph paper provided.
- Write down the values for l_{0R} and l_{0L}, with uncertainties.

Task 1.5 Calculating λ. Write down the calculated value for λ. Include its uncertainty and the analysis to obtain it. **SUGGESTION:** In your formula for λ, wherever you find $(L_b - L_a)$ replace it by d and use its measured value. **(2 points).**

EXPERIMENTAL PROBLEM 2
BIREFRINGENCE OF MICA

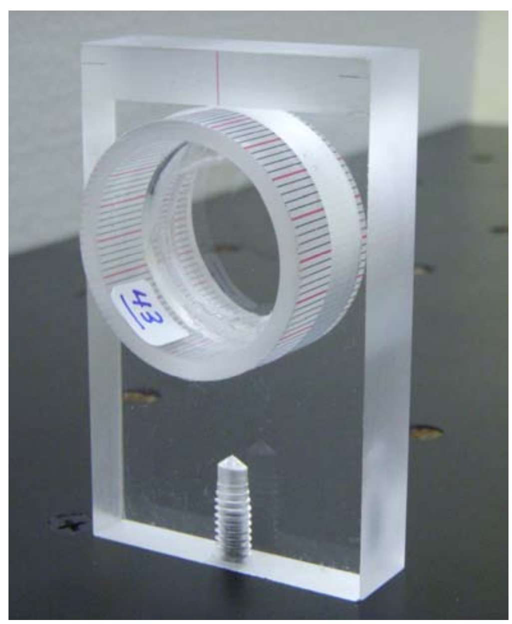
In this experiment you will measure the birefringence of mica (a crystal widely used in polarizing optical components).

MATERIAL

In addition to items 1), 2) and 3), you should use,

14) Two polarizing films mounted in slide holders, each with an additional acrylic support (LABEL J). See photograph for mounting instructions.
15) A thin mica plate mounted in a plastic cylinder with a scale with no numbers; acrylic support for the cylinder (LABEL K). See photograph for mounting instructions.
16) Photodetector equipment. A photodetector in a plastic box, connectors and foam support. A multimeter to measure the voltage of the photodetector (LABEL L). See photograph for mounting and connecting instructions.
17) Calculator.
18) White index cards, masking tape, stickers, scissors, triangle squares set.
19) Pencils, paper, graph paper.

Polarizer mounted in slide holder with acrylic support (LABEL J).



Thin mica plate mounted in cylinder with a scale with no numbers, and acrylic support (LABEL K).

A photodetector in a plastic box, connectors and foam support. A multimeter to measure the voltage of the photodetector (LABEL L). Set the connections as indicated.

DESCRIPTION OF THE PHENOMENON

Light is a transverse electromagnetic wave, with its electric field lying on a plane perpendicular to the propagation direction and oscillating in time as the light wave travels.

If the direction of the electric field remains in time oscillating *along a single line*, the wave is said to be linearly polarized, or simply, polarized. See Figure 2.1.

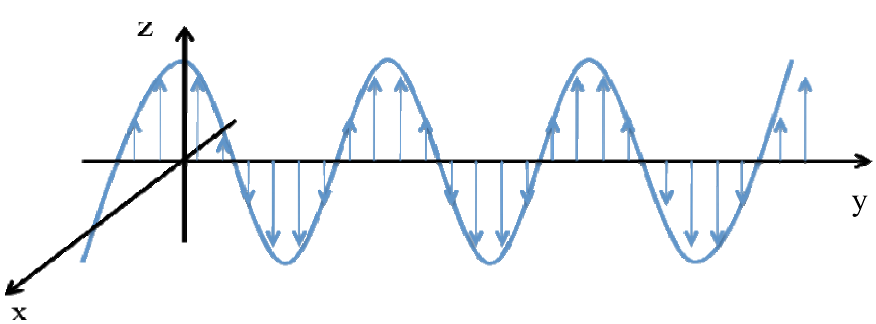


Figure 2.1 A wave travelling in the y-direction and polarized in the z-direction.

A polarizing film (or simply, a polarizer) is a material with a privileged axis parallel to its surface, such that, transmitted light emerges polarized along the axis of the polarizer. Call (+) the privileged axis and (-) the perpendicular one.

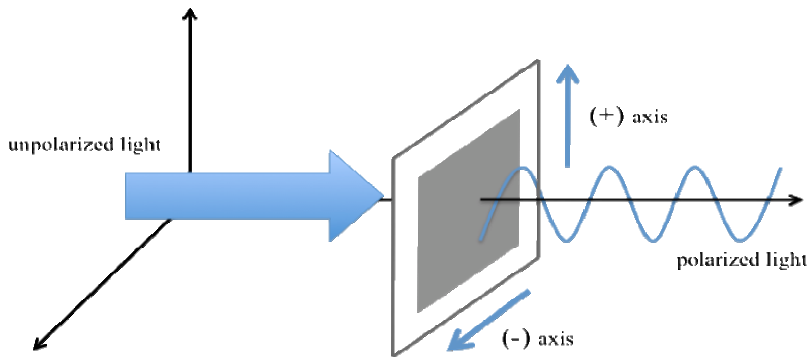


Figure 2.2 Unpolarized light normally incident on a polarizer. Transmitted light is polarized in the (+) direction of the polarizer.

Common transparent materials (such as window glass), transmit light with the same polarization as the incident one, because its index of refraction does not depend on the direction and/or polarization of the incident wave. Many crystals, including mica, however, are sensitive to the direction of the electric field of the wave. For propagation perpendicular to its surface, the mica sheet has two characteristic orthogonal axes, which we will call Axis 1 and Axis 2. This leads to the phenomenon called birefringence.

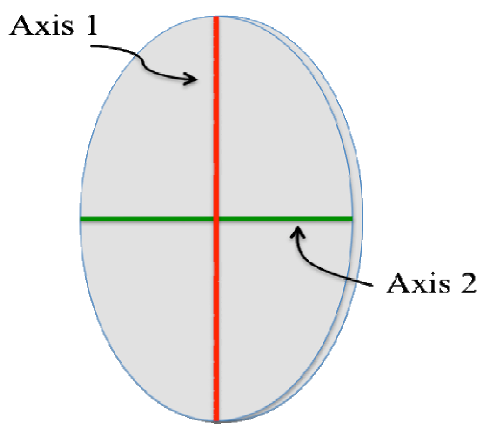


Figure 2.3 Thin slab of mica with its two axes, Axis 1 (red) and Axis 2 (green).

Let us analyze two simple cases to exemplify the birefringence. Assume that a wave **polarized in the vertical direction** is normally incident on one of the surfaces of the thin slab of mica.

Case 1) Axis 1 or Axis 2 is parallel to the polarization of the incident wave. The trasmitted wave passes without changing its polarization state, but the propagation is characterized as if the material had either an index of refraction n_1 or n_2. See Figs. 2.4 and 2.5.

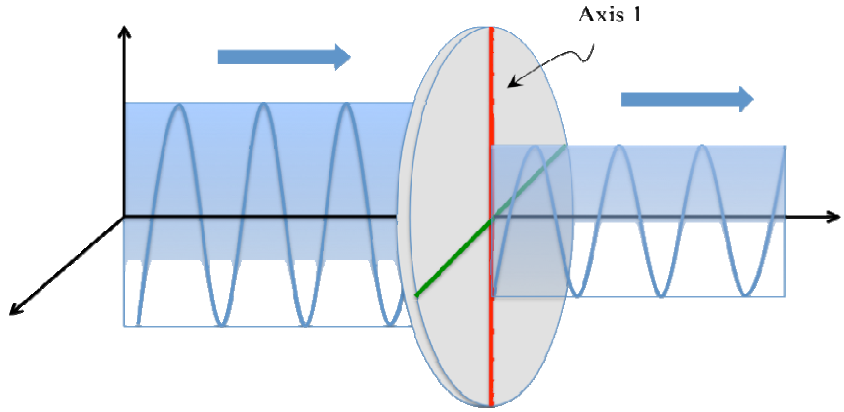


Figure 2.4 Axis 1 is parallel to polarization of incident wave. Index of refraction is n_1.

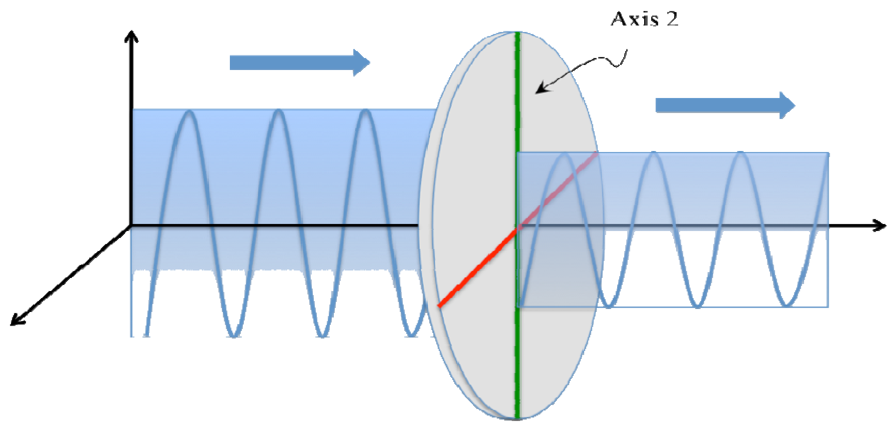


Figure 2.5 Axis 2 is parallel to polarization of incident wave. Index of refraction is n_2.

Case 2) Axis 1 makes an angle θ with the direction of polarization of the incident wave. The transmitted light has a more complicated polarization state. This wave, however, can be seen as the *superposition* of two waves with different phases, one that has polarization **parallel** to the polarization of the incident wave (i.e. "vertical") and another that has polarization **perpendicular** to the polarization of the incident wave (i.e. "horizontal").

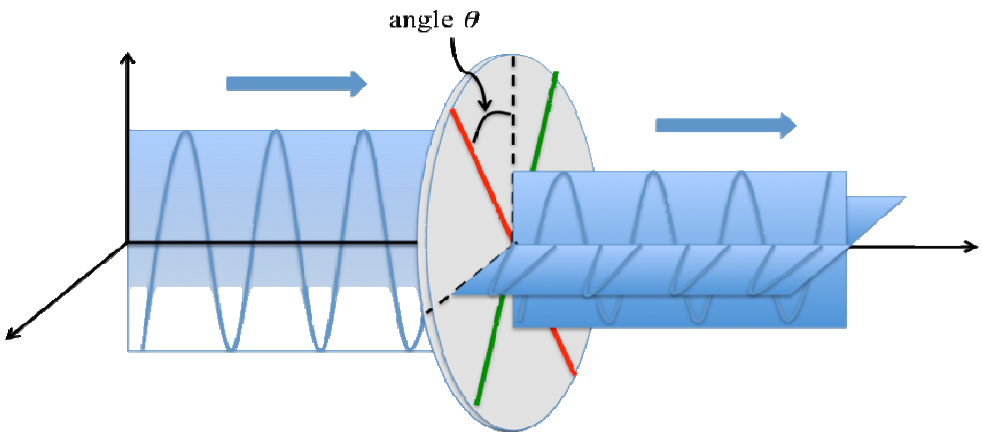


Figure 2.6 Axis 1 makes and angle θ with polarization of incident wave

Call I_P the *intensity* of the wave transmitted *parallel* to the polarization of the incident wave, and I_O the *intensity* of the wave transmitted *perpendicular* to polarization of the incident wave. These intensities depend on the angle θ, on the wavelength λ of the light source, on the thickness L of the thin plate, and on the absolute value of the difference of the refractive indices, $|n_1 - n_2|$. This last quantity is called the *birefringence* of the material. The measurement of this quantity is the goal of this problem. Together with polarizers, birefringent materials are useful for the control of light polarization states.

We point out here that the photodetector measures the intensity of the light incident on it, independent of its polarization.

The dependence of $I_P(\theta)$ and $I_O(\theta)$ on the angle θ is complicated due to other effects not considered, such as the absorption of the incident radiation by the mica. One can obtain, however, approximated but very simple expressions for the normalized intensities $\bar{I}_P(\theta)$ and $\bar{I}_O(\theta)$, defined as,

$$\bar{I}_P(\theta) = \frac{I_P(\theta)}{I_P(\theta) + I_O(\theta)} \tag{2.1}$$

and

$$\bar{I}_O(\theta) = \frac{I_O(\theta)}{I_P(\theta) + I_O(\theta)} \tag{2.2}$$

It can be shown that the normalized intensities are (approximately) given by,

$$\bar{I}_P(\theta) = 1 - \frac{1}{2}(1 - \cos\Delta\phi)\sin^2(2\theta) \tag{2.3}$$

and

$$\bar{I}_O(\theta) = \frac{1}{2}(1 - \cos\Delta\phi)\sin^2(2\theta) \qquad (2.4)$$

where $\Delta\phi$ is the difference of phases of the parallel and perpendicular transmitted waves. This quantity is given by,

$$\Delta\phi = \frac{2\pi L}{\lambda}|n_1 - n_2| \qquad (2.5)$$

where L is the thickness of the thin plate of mica, λ the wavelength of the incident radiation and $|n_1 - n_2|$ the birefringence.

EXPERIMENTAL SETUP

Task 2.1 Experimental setup for measuring intensities. Design an experimental setup for measuring the intensities I_P and I_O of the transmitted wave, as a function of the angle θ of any of the optical axes, as shown in Fig. 2.6. *Do this by writing the LABELS of the different devices on the drawing of the optical table.* Use the convention (+) and (-) for the direction of the polarizers. You can make additional simple drawings to help clarify your design.

Task 2.1 a) Setup for I_P **(0.5 points).**
Task 2.1 b) Setup for I_O **(0.5 points).**

Laser beam alignment. Align the laser beam in such a way that it is parallel to the table and is incident on the center of the cylinder holding the mica. You may align by using one the white index cards to follow the path. Small adjustments can be made with the movable mirror.

Photodetector and the multimeter. The photodetector produces a voltage as light impinges on it. Measure this voltage with the multimeter provided. The voltage produced is linearly proportional to the intensity of the light. Thus, report the intensities as the voltage produced by the photodetector. Without any laser beam incident on the photodetector, you can measure the background light intensity of the detector. This should be less than 1 mV. *Do not correct* for this background when you perform the intensity measurements.

WARNING: The laser beam is partially polarized but it is not known in which direction. Thus, to obtain polarized light with good intensity readings, place a polarizer with either its (+) or (–) axes vertically in such a way that you obtain the maximum transmitted intensity in the absence of any other optical device.

MEASURING INTENSITIES

Task 2.2 The scale for angle settings. The cylinder holding the mica has a regular graduation for settings of the angles. Write down the value in degrees of the smallest interval (*i.e.* between two black consecutive lines). **(0.25 points).**

Finding (approximately) the zero of θ and/or the location of the mica axes. To facilitate the analysis, it is very important that you find the appropriate zero of the angles. We suggest that, first, you identify the location of one of the mica axes, and call it Axis 1. It is almost sure that this position will not coincide with a graduation line on the cylinder. Thus, consider the nearest graduation line in the mica cylinder as the provisional origin for the angles. Call $\bar{\theta}$ the angles measured from such an origin. Below you will be asked to provide a more accurate location of the zero of θ.

Task 2.3 Measuring I_P and I_O. Measure the intensities I_P and I_O for as many angles $\bar{\theta}$ as you consider necessary. Report your measurements in Table I. Try to make the measurements for I_P and I_O for the *same* setting of the cylinder with the mica, that is, for a *fixed* angle $\bar{\theta}$. **(3.0 points).**

Task 2.4 Finding an appropriate zero for θ. The location of Axis 1 defines the zero of the angle θ. As mentioned above, it is mostly sure that the location of Axis 1 does not coincide with a graduation line on the mica cylinder. To find the zero of the angles, you may proceed either graphically or numerically. Recognize that the relationship near a maximum or a minimum may be approximated by a parabola where:

$$I(\bar{\theta}) \approx a\bar{\theta}^2 + b\bar{\theta} + c$$

and the minimum or maximum of the parabola is given by,

$$\bar{\theta}_m = -\frac{b}{2a}.$$

Either of the above choices gives rise to a shift $\delta\bar{\theta}$ of all your values of $\bar{\theta}$ given in Table I of Task 2.3, such that they can now be written as angles θ from the appropriate zero, $\theta = \bar{\theta} + \delta\bar{\theta}$. Write down the value of the shift $\delta\bar{\theta}$ in degrees. **(1.0 points).**

DATA ANALYSIS.

Task 2.5 Choosing the appropriate variables. Choose $\bar{I}_P(\theta)$ or $\bar{I}_O(\theta)$ to make an analysis to find the difference of phases $\Delta\phi$. Identify the variables that you will use. (**0.5 point).**

Task 2.6 Data analysis and the phase difference.

- Use Table II to write down the values of the variables needed for their analysis. Make sure that you use the corrected values for the angles θ. Include uncertainties. Use graph paper to plot your variables. **(1.0 points).**

- Perform an analysis of the data needed to obtain the phase difference $\Delta\phi$. Report your results including uncertainties. Write down any equations or formulas used in the analysis. Plot your results. **(1.75 points).**

- Calculate the value of the phase difference $\Delta\phi$ in radians, including its uncertainty. Find the value of the phase difference in the interval $[0,\pi]$. **(0.5 points).**

Task 2.7 Calculating the birefringence $|n_1 - n_2|$. You may note that if you add $2N\pi$ to the phase difference $\Delta\phi$, with *N* any integer, or if you change the sign of the phase, the values of the intensities are unchanged. However, the value of the birefringence $|n_1 - n_2|$ would change. Thus, to use the value $\Delta\phi$ found in Task 2.6 to correctly calculate the birefringence, you must consider the following:

$$\Delta\phi = \frac{2\pi L}{\lambda}|n_1 - n_2| \qquad \text{if} \qquad L < 82\times10^{-6}\ \text{m}$$

or

$$2\pi - \Delta\phi = \frac{2\pi L}{\lambda}|n_1 - n_2| \qquad \text{if} \qquad L > 82\times10^{-6}\ \text{m}$$

where the value L of the thickness of the slab of mica you used is written on the cylinder holding it. This number is given in micrometers (1 micrometer = 10^{-6} m). Assign 1×10^{-6}m as the uncertainty for L. For the laser wavelength, you may use the value you found in Problem 1 or the average value between 620×10^{-9} m and 750×10^{-9} m, the reported range for red in the visible spectrum. Write down the values of L and λ as well as the birefringence $|n_1 - n_2|$ with its uncertainty. Include the formulas that you used to calculate the uncertainties. **(1.0 points).**

THEORETICAL PROBLEM No. 1

EVOLUTION OF THE EARTH-MOON SYSTEM

Scientists can determine the distance Earth-Moon with great precision. They achieve this by bouncing a laser beam on special mirrors deposited on the Moon´s surface by astronauts in 1969, and measuring the round travel time of the light (see Figure 1).

Figure 1. A laser beam sent from an observatory is used to measure accurately the distance between the Earth and the Moon.

With these observations, they have directly measured that the Moon is slowly receding from the Earth. That is, the Earth-Moon distance is increasing with time. This is happening because due to tidal torques the Earth is transferring angular momentum to the Moon, see Figure 2. In this problem you will derive the basic parameters of the phenomenon.

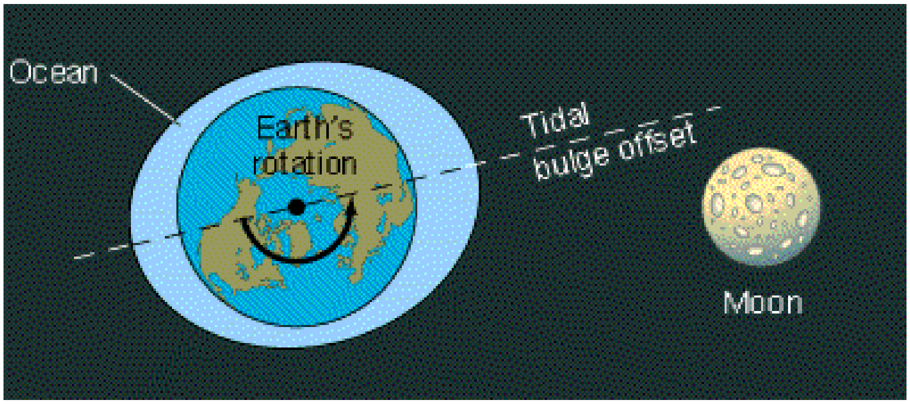


Figure 2. The Moon's gravity produces tidal deformations or "bulges" in the Earth. Because of the Earth's rotation, the line that goes through the bulges is not aligned with the line between the Earth and the Moon. This misalignment produces a torque that transfers angular momentum from the Earth's rotation to the Moon's translation. The drawing is not to scale.

1. Conservation of Angular Momentum.

Let L_1 be the present total angular momentum of the Earth-Moon system. Now, make the following assumptions: i) L_1 is the sum of the rotation of the Earth around its axis and the translation of the Moon in its orbit around the Earth only. ii) The Moon's orbit is circular and the Moon can be taken as a point. iii) The Earth's axis of rotation and the Moon's axis of revolution are parallel. iv) To simplify the calculations, we take the motion to be around the center of the Earth and not the center of mass. Throughout the problem, all moments of inertia, torques and angular momenta are defined around the axis of the Earth. v) Ignore the influence of the Sun.

1a	Write down the equation for the present total angular momentum of the Earth-Moon system. Set this equation in terms of I_E, the moment of inertia of the Earth; ω_{E1}, the present angular frequency of the Earth's rotation; I_{M1}, the present moment of inertia of the Moon with respect to the Earth´s axis; and ω_{M1}, the present angular frequency of the Moon's orbit.	0.2

This process of transfer of angular momentum will end when the period of rotation of the Earth and the period of revolution of the Moon around the Earth have the same duration. At this point the tidal bulges produced by the Moon on the Earth will be aligned with the line between the Moon and the Earth and the torque will disappear.

1b	Write down the equation for the final total angular momentum L_2 of the Earth-Moon system. Make the same assumptions as in Question 1a. Set this equation in terms of I_E, the moment of inertia of the Earth; ω_2, the final angular frequency of the Earth's rotation and Moon's translation; and I_{M2}, the final moment of inertia of the Moon.	0.2

1c	Neglecting the contribution of the Earth´s rotation to the final total angular momentum, write down the equation that expresses the angular momentum conservation for this problem.	0.3

2. Final Separation and Final Angular Frequency of the Earth-Moon System.

Assume that the gravitational equation for a circular orbit (of the Moon around the Earth) is always valid. Neglect the contribution of the Earth´s rotation to the final total angular momentum.

2a	Write down the gravitational equation for the circular orbit of the Moon around the Earth, at the final state, in terms of M_E, ω_2, G and the final separation D_2 between the Earth and the Moon. M_E is the mass of the Earth and G is the gravitational constant.	0.2

2b	Write down the equation for the final separation D_2 between the Earth and the Moon in terms of the known parameters, L_1, the total angular momentum of the system, M_E and M_M, the masses of the Earth and Moon, respectively, and G.	0.5

2c	Write down the equation for the final angular frequency ω_2 of the Earth-Moon system in terms of the known parameters L_1, M_E, M_M and G.	0.5

Below you will be asked to find the numerical values of D_2 and ω_2. For this you need to know the moment of inertia of the Earth.

2d	Write down the equation for the moment of inertia of the Earth I_E assuming it is a sphere with inner density ρ_i from the center to a radius r_i, and with outer density ρ_o from the radius r_i to the surface at a radius r_o (see Figure 3).	0.5

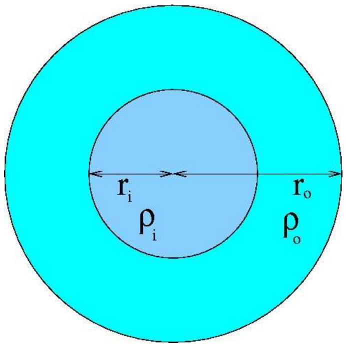


Figure 3. The Earth as a sphere with two densities, ρ_i and ρ_o.

Determine the numerical values requested in this problem always to *two significant digits*.

2e	Evaluate the moment of inertia of the Earth I_E, using $\rho_i = 1.3\times10^4$ kg m^{-3}, $r_i = 3.5\times10^6$ m, $\rho_o = 4.0\times10^3$ kg m^{-3}, and $r_o = 6.4\times10^6$ m.	0.2

The masses of the Earth and Moon are $M_E = 6.0\times10^{24}$ kg and $M_M = 7.3\times10^{22}$ kg, respectively. The present separation between the Earth and the Moon is $D_1 = 3.8\times10^8$ m. The present angular frequency of the Earth's rotation is $\omega_{E1} = 7.3\times10^{-5}$ s^{-1}. The present angular frequency of the Moon's translation around the Earth is $\omega_{M1} = 2.7\times10^{-6}$ s^{-1}, and the gravitational constant is $G = 6.7\times10^{-11}$ m^3 kg^{-1} s^{-2}.

2f	Evaluate the numerical value of the total angular momentum of the system, L_1.	0.2

2g	Find the final separation D_2 in meters and in units of the present separation D_1.	0.3

2h	Find the final angular frequency ω_2 in s^{-1}, as well as the final duration of the day in units of present days.	0.3

Verify that the assumption of neglecting the contribution of the Earth´s rotation to the final total angular momentum is justified by finding the ratio of the final angular momentum of the Earth to that of the Moon. This should be a small quantity.

2i	Find the ratio of the final angular momentum of the Earth to that of the Moon.	0.2

3. How much is the Moon receding per year?

Now, you will find how much the Moon is receding from the Earth each year. For this, you will need to know the equation for the torque acting at present on the Moon. Assume that the tidal bulges can be approximated by two point masses, each of mass m, located on the surface of the Earth, see Fig. 4. Let θ be the angle between the line that goes through the bulges and the line that joins the centers of the Earth and the Moon.

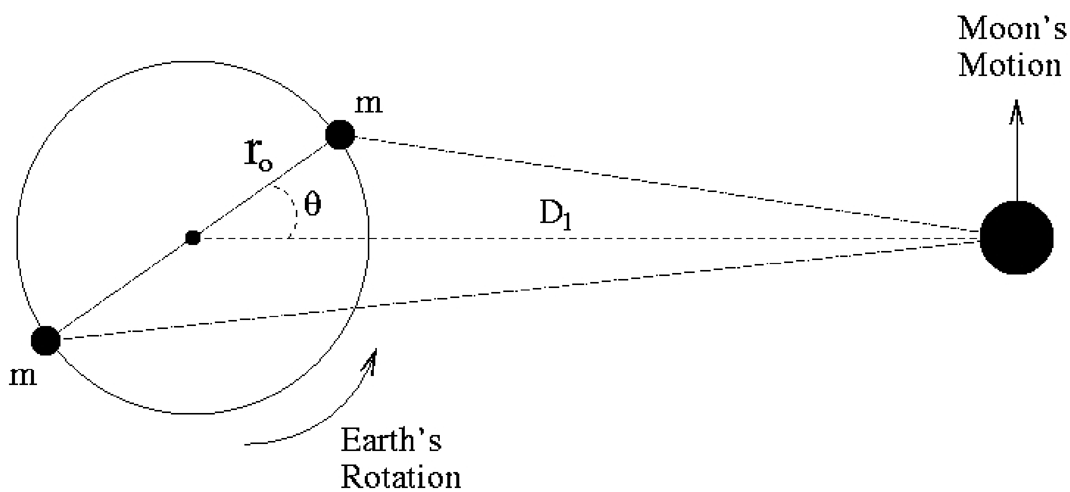


Figure 4. Schematic diagram to estimate the torque produced on the Moon by the bulges on the Earth. The drawing is not to scale.

3a	Find F_c, the magnitude of the force produced on the Moon by the closest point mass.	0.4

3b	Find F_f, the magnitude of the force produced on the Moon by the farthest point mass.	0.4

You may now evaluate the torques produced by the point masses.

3c	Find the magnitude of τ_c, the torque produced by the closest point mass.	0.4

3d	Find the magnitude of τ_f, the torque produced by the farthest point mass.	0.4

3e	Find the magnitude of the total torque τ produced by the two masses. Since $r_o << D_1$ you should approximate your expression to lowest significant order in r_o / D_1. You may use that $(1+x)^a \approx 1+ax$, if $x<<1$.	1.0

3f	Calculate the numerical value of the total torque τ, taking into account that $\theta = 3^\circ$ and that $m = 3.6\times10^{16}$ kg (note that this mass is of the order of 10^{-8} times the mass of the Earth).	0.5

Since the torque is the rate of change of angular momentum with time, find the increase in the distance Earth-Moon at present, per year. For this step, express the angular momentum of the Moon in terms of M_M, M_E, D_1 and G only.

3g	Find the increase in the distance Earth-Moon at present, per year.	1.0

Finally, estimate how much the length of the day is increasing each year.

3h	Find the decrease of ω_{E1} per year and how much is the length of the day at present increasing each year.	1.0

4. Where is the energy going?

In contrast to the angular momentum, that is conserved, the total (rotational plus gravitational) energy of the system is not. We will look into this in this last section.

4a	Write down an equation for the total (rotational plus gravitational) energy of the Earth-Moon system at present, E. Put this equation in terms of I_E, ω_{E1}, M_M, M_E, D_1 and G only.	0.4

4b	Write down an equation for the change in E, ΔE, as a function of the changes in D_1 and in ω_{E1}. Evaluate the numerical value of ΔE for a year, using the values of changes in D_1 and in ω_{E1} found in questions 3g and 3h.	0.4

Verify that this loss of energy is consistent with an estimate for the energy dissipated as heat in the tides produced by the Moon on the Earth. Assume that the tides rise, on the average by 0.5 m, a layer of water $h =$ 0.5 m deep that covers the surface of the Earth (for simplicity assume that all the surface of the Earth is covered with water). This happens twice a day. Further assume that 10% of this gravitational energy is dissipated as heat due to viscosity when the water descends. Take the density of water to be $\rho_{water} = 10^3$ kg m^{-3}, and the gravitational acceleration on the surface of the Earth to be $g = 9.8$ m s^{-2}.

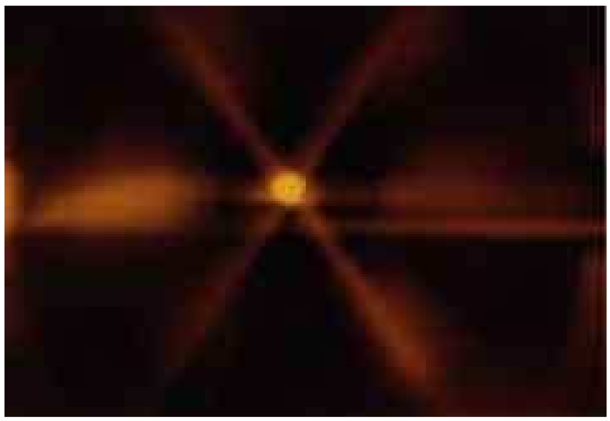
4c	What is the mass of this surface layer of water?	0.2

4d	Calculate how much energy is dissipated in a year? How does this compare with the energy lost per year by the Earth-Moon system at present?	0.3

THEORETICAL PROBLEM 2

DOPPLER LASER COOLING AND OPTICAL MOLASSES

The purpose of this problem is to develop a simple theory to understand the so-called "laser cooling" and "optical molasses" phenomena. This refers to the cooling of a beam of neutral atoms, typically alkaline, by counterpropagating laser beams with the same frequency. This is part of the Physics Nobel Prize awarded to S. Chu, P. Phillips and C. Cohen-Tannoudji in 1997.



The image above shows sodium atoms (the bright spot in the center) trapped at the intersection of three orthogonal pairs of opposing laser beams. The trapping region is called "optical molasses" because the dissipative optical force resembles the viscous drag on a body moving through molasses.

In this problem you will analyze the basic phenomenon of the interaction between a photon incident on an atom and the basis of the dissipative mechanism in one dimension.

PART I: BASICS OF LASER COOLING

Consider an atom of mass m moving in the $+x$ direction with velocity v. For simplicity, we shall consider the problem to be one-dimensional, namely, we shall ignore the y and z directions (see figure 1). The atom has two internal energy levels. The energy of the lowest state is considered to be zero and the energy of the excited state to be $\hbar\omega_0$, where $\hbar = h/2\pi$. The atom is initially in the lowest state. A laser beam with frequency ω_L in the laboratory is directed in the $-x$ direction and it is incident on the atom. Quantum mechanically the laser is composed of a large number of photons, each with energy $\hbar\omega_L$ and momentum $-\hbar q$. A photon can be absorbed by the atom and later spontaneously emitted; this emission can occur with equal probabilities along the $+x$ and $-x$ directions. Since the atom moves at non-relativistic speeds, $v/c << 1$ (with c the speed of light) keep terms up to first order in this quantity only. Consider also $\hbar q/mv << 1$, namely, that the momentum of the atom is much larger than the momentum of a single photon. In writing your answers, keep only corrections linear in either of the above quantities.

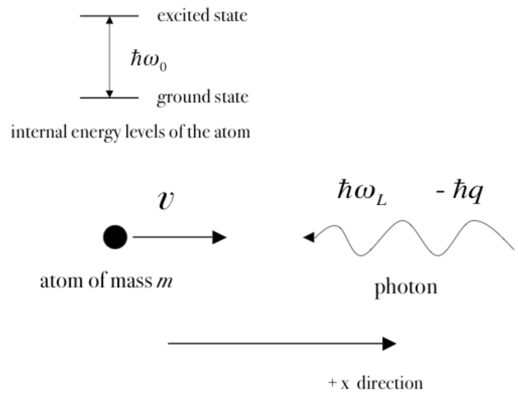


Fig.1 Sketch of an atom of mass m with velocity v in the $+x$ direction, colliding with a photon with energy $\hbar\omega_L$ and momentum $-\hbar q$. The atom has two internal states with energy difference $\hbar\omega_0$.

Assume that the laser frequency ω_L is tuned such that, as seen by the moving atom, it is in resonance with the internal transition of the atom. Answer the following questions:

1. Absorption.

1a	Write down the resonance condition for the absorption of the photon.	0.2

1b	Write down the momentum p_{at} of the atom after absorption, as seen in the laboratory.	0.2

1c	Write down the total energy ε_{at} of the atom after absorption, as seen in the laboratory.	0.2

2. Spontaneous emission of a photon in the $-x$ direction.

At some time after the absorption of the incident photon, the atom may emit a photon in the $-x$ direction.

2a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

2b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

2c	Write down the momentum of the atom p_{at}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

2d	Write down the total energy of the atom ε_{at}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

3. Spontaneous emission of a photon in the $+x$ direction.

At some time after the absorption of the incident photon, the atom may instead emit a photon in the $+x$ direction.

3a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

3b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

3c	Write down the momentum of the atom p_{at}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

3d	Write down the total energy of the atom ε_{at}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

4. Average emission after the absorption.

The spontaneous emission of a photon in the $-x$ or in the $+x$ directions occurs with the same probability. Taking this into account, answer the following questions.

4a	Write down the average energy of an emitted photon, ε_{ph}, after the emission process.	0.2

4b	Write down the average momentum of an emitted photon p_{ph}, after the emission process.	0.2

4c	Write down the average total energy of the atom ε_{at}, after the emission process.	0.2

4d	Write down the average momentum of the atom p_{at}, after the emission process.	0.2

5. Energy and momentum transfer.

Assuming a complete one-photon absorption-emission process only, as described above, there is a net average momentum and energy transfer between the laser radiation and the atom.

5a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process.	0.2

5b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process.	0.2

6. Energy and momentum transfer by a laser beam along the $+x$ direction.

Consider now that a laser beam of frequency ω_L' is incident on the atom along the $+x$ direction, while the atom moves also in the $+x$ direction with velocity v. Assuming a resonance condition between the internal transition of the atom and the laser beam, as seen by the atom, answer the following questions:

6a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process.	0.3

6b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process.	0.3

PART II: DISSIPATION AND THE FUNDAMENTALS OF OPTICAL MOLASSES

Nature, however, imposes an inherent uncertainty in quantum processes. Thus, the fact that the atom can spontaneously emit a photon in a *finite* time after absorption, gives as a result that the resonance condition does not have to be obeyed *exactly* as in the discussion above. That is, the frequency of the laser beams ω_L and ω_L' may have any value and the absorption-emission process can still occur. These will happen with different (quantum) probabilities and, as one should expect, the maximum probability is found at the exact resonance condition. On the average, the time elapsed between a single process of absorption and emission is called the lifetime of the excited energy level of the atom and it is denoted by Γ^{-1}.

Consider a collection of N atoms at *rest* in the laboratory frame of reference, and a laser beam of frequency ω_L incident on them. The atoms absorb and emit continuously such that there is, on average, N_{exc} atoms in the excited state (and therefore, $N - N_{exc}$ atoms in the ground state). A quantum mechanical calculation yields the following result:

$$N_{exc} = N \frac{\Omega_R^2}{(\omega_0 - \omega_L)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2}$$

where ω_0 is the resonance frequency of the atomic transition and Ω_R is the so-called Rabi frequency; Ω_R^2 is proportional to the *intensity* of the laser beam. As mentioned above, you can see that this number is different from zero even if the resonance frequency ω_0 is different from the frequency of the laser beam ω_L. An alternative way of expressing the previous result is that the number of absorption-emission processes per unit of time is $N_{exc}\Gamma$.

Consider the physical situation depicted in Figure 2, in which two counter propagating laser beams with the *same* but *arbitrary* frequency ω_L are incident on a gas of N atoms that move in the $+x$ direction with velocity v.

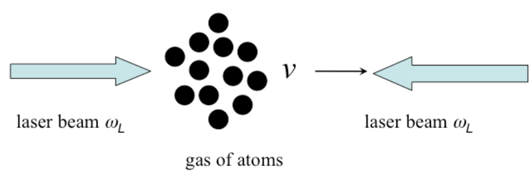


Figure 2. Two counter propagating laser beams with the *same* but *arbitrary* frequency ω_L are incident on a gas of N atoms that move in the $+x$ direction with velocity v.

7. Force on the atomic beam by the lasers.

7a	With the information found so far, find the force that the lasers exert on the atomic beam. You should assume that $mv >> \hbar q$.	1.5

8. Low velocity limit.

Assume now that the velocity of the atoms is small enough, such that you can expand the force up to first order in v.

8a	Find an expression for the force found in Question (7a), in this limit.	1.5

Using this result, you can find the conditions for speeding up, slowing down, or no effect at all on the atoms by the laser radiation.

8b	Write down the condition to obtain a positive force (speeding up the atoms).	0.25

8c	Write down the condition to obtain a zero force.	0.25

8d	Write down the condition to obtain a negative force (slowing down the atoms).	0.25

8e	Consider now that the atoms are moving with a velocity $-v$ (in the $-x$ direction). Write down the condition to obtain a slowing down force on the atoms.	0.25

9. Optical molasses.

In the case of a negative force, one obtains a frictional dissipative force. Assume that initially, at $t=0$, the gas of atoms has velocity v_0.

9a	In the limit of low velocities, find the velocity of the atoms after the laser beams have been on for a time τ.	1.5

9b	Assume now that the gas of atoms is in thermal equilibrium at a temperature T_0. Find the temperature T after the laser beams have been on for a time τ.	0.5

This model does not allow you to go to arbitrarily low temperatures.

THEORETICAL PROBLEM No. 3

WHY ARE STARS SO LARGE?

The stars are spheres of hot gas. Most of them shine because they are fusing hydrogen into helium in their central parts. In this problem we use concepts of both classical and quantum mechanics, as well as of electrostatics and thermodynamics, to understand why stars have to be big enough to achieve this fusion process and also derive what would be the mass and radius of the smallest star that can fuse hydrogen.

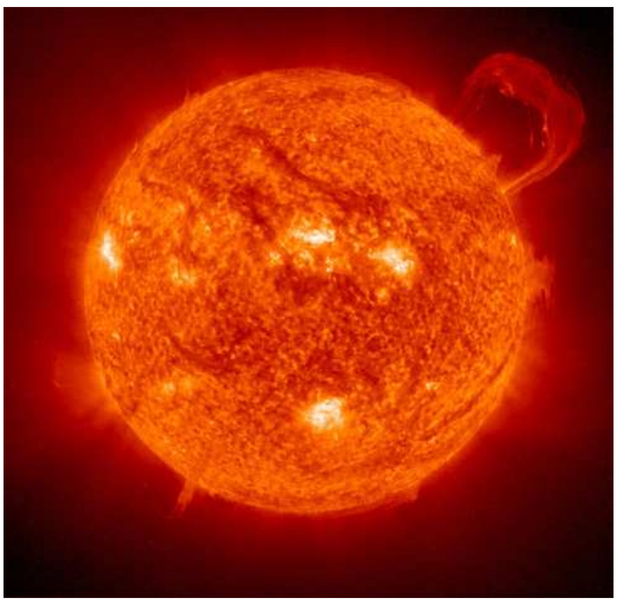


Figure 1. Our Sun, as most stars, shines as a result of thermonuclear fusion of hydrogen into helium in its central parts.

USEFUL CONSTANTS

Gravitational constant = $G = 6.7 \times 10^{-11}$ m^3 kg^{-1} s^2

Boltzmann´s constant = $k = 1.4 \times 10^{-23}$ J K^{-1}

Planck's constant = $h = 6.6 \times 10^{-34}$ m^2 kg s^{-1}

Mass of the proton = $m_p = 1.7 \times 10^{-27}$ kg

Mass of the electron = $m_e = 9.1 \times 10^{-31}$ kg

Unit of electric charge = $q = 1.6 \times 10^{-19}$ C

Electric constant (vacuum permittivity) = $\varepsilon_0 = 8.9 \times 10^{-12}$ C^2 N^{-1} m^{-2}

Radius of the Sun = $R_S = 7.0 \times 10^8$ m

Mass of the Sun = $M_S = 2.0 \times 10^{30}$ kg

1. A classical estimate of the temperature at the center of the stars.

Assume that the gas that forms the star is pure ionized hydrogen (electrons and protons in equal amounts), and that it behaves like an ideal gas. From the classical point of view, to fuse two protons, they need to get as close as 10^{-15} m for the short range strong nuclear force, which is attractive, to become dominant. However, to bring them together they have to overcome first the repulsive action of Coulomb's force. Assume classically that the two protons (taken to be point sources) are moving in an antiparallel way, each with velocity v_{rms}, the root-mean-square (rms) velocity of the protons, in a one-dimensional frontal collision.

1a	What has to be the temperature of the gas, T_c, so that the distance of closest approach of the protons, d_c, equals 10^{-15} m? Give this and all numerical values in this problem up to two significant figures.	1.5

2. Finding that the previous temperature estimate is wrong.

To check if the previous temperature estimate is reasonable, one needs an independent way of estimating the central temperature of a star. The structure of the stars is very complicated, but we can gain significant understanding making some assumptions. Stars are in equilibrium, that is, they do not expand or contract because the inward force of gravity is balanced by the outward force of pressure (see Figure 2). For a slab of gas the equation of hydrostatic equilibrium at a given distance r from the center of the star, is given by

$$\frac{\Delta P}{\Delta r} = -\frac{G M_r \rho_r}{r^2},$$

where P is the pressure of the gas, G the gravitational constant, M_r the mass of the star within a sphere of radius r, and ρ_r is the density of the gas in the slab.

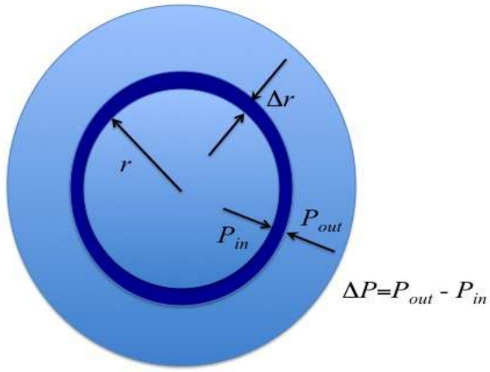


Figure 2. The stars are in hydrostatic equilibrium, with the pressure difference balancing gravity.

An order of magnitude estimate of the central temperature of the star can be obtained with values of the parameters at the center and at the surface of the star, making the following approximations:

$\Delta P \approx P_o - P_c$,

where P_c and P_o are the pressures at the center and surface of the star, respectively. Since $P_c >> P_o$, we can assume that

$\Delta P \approx -P_c$.

Within the same approximation, we can write

$\Delta r \approx R$,

where R is the total radius of the star, and

$M_r \approx M_R = M$,

with M the total mass of the star.

The density may be approximated by its value at the center,

$\rho_r \approx \rho_c$.

You can assume that the pressure is that of an ideal gas.

2a	Find an equation for the temperature at the center of the star, T_c, in terms of the radius and mass of the star and of physical constants only.	0.5

We can use now the following prediction of this model as a criterion for its validity:

2b	Using the equation found in (2a) write down the ratio M/R expected for a star in terms of physical constants and T_c only.	0.5

2c	Use the value of T_c derived in section (1a) and find the numerical value of the ratio M/R expected for a star.	0.5

2d	Now, calculate the ratio $M(Sun)/R(Sun)$, and verify that this value is much smaller than the one found in (2c).	0.5

3. A quantum mechanical estimate of the temperature at the center of the stars

The large discrepancy found in (2d) suggests that the classical estimate for T_c obtained in (1a) is not correct. The solution to this discrepancy is found when we consider quantum mechanical effects, that tell us that the protons behave as waves and that a single proton is smeared on a size of the order of λ_p, the de Broglie wavelength. This implies that if d_c, the distance of closest approach of the protons is of the order of λ_p, the protons in a quantum mechanical sense overlap and can fuse.

3a	Assuming that $d_c = \frac{\lambda_p}{2^{1/2}}$ is the condition that allows fusion, for a proton with velocity v_{rms}, find an equation for T_c in terms of physical constants only.	1.0

3b	Evaluate numerically the value of T_c obtained in (3a).	0.5

3c	Use the value of T_c derived in (3b) to find the numerical value of the ratio M/R expected for a star, using the formula derived in (2b). Verify that this value is quite similar to the ratio $M(Sun)/R(Sun)$ observed.	0.5

Indeed, stars in the so-called *main sequence* (fusing hydrogen) approximately do follow this ratio for a large range of masses.

4. The mass/radius ratio of the stars.

The previous agreement suggests that the quantum mechanical approach for estimating the temperature at the center of the Sun is correct.

4a	Use the previous results to demonstrate that for any star fusing hydrogen, the ratio of mass M to radius R is the same and depends only on physical constants. Find the equation for the ratio M/R for stars fusing hydrogen.	0.5

5. The mass and radius of the smallest star.

The result found in (4a) suggests that there could be stars of any mass as long as such a relationship is fulfilled; however, this is not true.

The gas inside normal stars fusing hydrogen is known to behave approximately as an ideal gas. This means that d_e, the typical separation *between electrons* is on the average larger that λ_e, their typical de Broglie wavelength. If closer, the electrons would be in a so-called degenerate state and the stars would behave differently. Note the distinction in the ways we treat protons and electrons inside the star. For protons, their de Broglie waves should overlap closely as they collide in order to fuse, whereas for electrons their de Broglie waves should not overlap in order to remain as an ideal gas.

The density in the stars increases with decreasing radius. Nevertheless, for this order-of-magnitude estimate assume they are of uniform density. You may further use that $m_p >> m_e$.

5a	Find an equation for n_e, the average electron number density inside the star.	0.5

5b	Find an equation for d_e, the typical separation between electrons inside the star.	0.5

5c	Use the $d_e \geq \frac{\lambda_e}{2^{1/2}}$ condition to write down an equation for the radius of the smallest normal star possible. Take the temperature at the center of the star as typical for all the stellar interior.	1.5

5d	Find the numerical value of the radius of the smallest normal star possible, both in meters and in units of solar radius.	0.5

5e	Find the numerical value of the mass of the smallest normal star possible, both in kg and in units of solar masses.	0.5

6. Fusing helium nuclei in older stars.

As stars get older they will have fused most of the hydrogen in their cores into helium (He), so they are forced to start fusing helium into heavier elements in order to continue shining. A helium nucleus has two protons and two neutrons, so it has twice the charge and approximately four times the mass of a proton. We saw before that $d_c = \frac{\lambda_p}{2^{1/2}}$ is the condition for the protons to fuse.

6a	Set the equivalent condition for helium nuclei and find $v_{rms}(He)$, the rms velocity of the helium nuclei and $T(He)$, the temperature needed for helium fusion.	0.5