Problem A1) Prove that (21n+4)/(14n+3) is irreducible for every natural number n.

پاسخ :

$3(14n+3) - 2(21n+4) = 1.$

Problem A2) For what real values of x is $\sqrt{x + \sqrt{2x-1}} + \sqrt{x - \sqrt{2x-1}} = A$, given (a) $A = \sqrt{2}$, (b) A = 1, (c) A = 2, where only non-negative real numbers are allowed in square roots and the root always denotes the non-negative root?

پاسخ :

(a) any x in the interval [1/2,1]; (b) no solutions; (c) x=3/2.

حل :

Note that we require $x \geq 1/2$ to avoid a negative sign under the inner square roots. Since $(x-1)^2 \geq 0$, we have $x \geq \sqrt{2x-1}$, so there is no difficulty with $\sqrt{x - \sqrt{2x-1}}$, provided that $x \geq 1/2$.

Squaring gives $2x + 2\sqrt{x^2-2x+1} = A^2$. Note that the square root is |x-1|, not simply (x-1). So we get finally $2x + 2|x-1| = A^2$. It is now easy to see that we get the solutions above.

Problem A3) Let a, b, c be real numbers. Given the equation for cos x:

$$a \cos^2 x + b \cos x + c = 0,$$

form a quadratic equation in cos 2x whose roots are the same values of x. Compare the equations in cos x and cos 2x for a=4, b=2, c=-1.

حل :

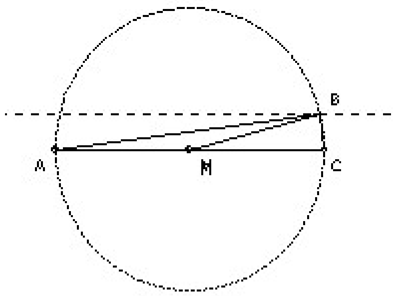
You need that $\cos 2x = 2 \cos^2 x - 1$. Some easy manipulation then gives:

$$a^2\cos^2 2x + (2a^2 + 4ac - 2b^2) \cos 2x + (4c^2 + 4ac - 2b^2 + a^2) = 0.$$

The equations are the same for the values of a, b, c given. The angles are $2\pi/5$ (or $8\pi/5$) and $4\pi/5$ (or $6\pi/5$).

Problem B1) Given the length |AC|, construct a triangle ABC with $\angle ABC = 90^o$, and the median BM satisfying $BM^2 = AB \cdot BC$.

حل :

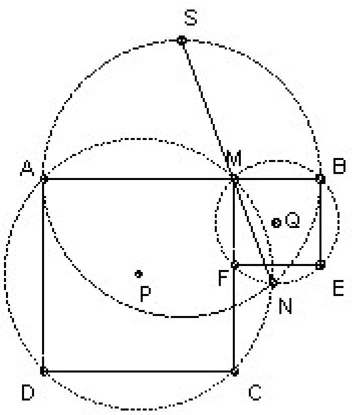


Area = $AB \cdot BC/2$ (because $\angle ABC = 90^o$= $BM^2/2$ (required) = $AC^2/8$ (because BM = AM = MC), so B lies a distance AC/4 from AC. Take B as the intersection of a circle diameter AC with a line parallel to AC distance AC/4.

Problem B2) An arbitrary point M is taken in the interior of the segment AB. Squares AMCD and MBEF are constructed on the same side of AB. The circles circumscribed about these squares, with centers P and Q, intersect at M and N.

(a) prove that AF and BC intersect at N;
(b) prove that the lines MN pass through a fixed point S (independent of M);
(c) find the locus of the midpoints of the segments PQ as M varies.

حل :



(a) $\angle ANM = \angle ACM = 45^o$. But $\angle FNM = \angle FEM = 45^o$, so A, F, N are collinear. Similarly, $\angle BNM = \angle BEM = 45^o$ and $\angle CNM = 180^o - \angle CAM = 135^o$, so B, N, C are collinear.

(b) Since $\angle ANM = \angle BNM = 45^o$, $\angle ANB = 90^o$, so N lies on the semicircle diameter AB. Let NM meet the circle diameter AB again at S. $\angle ANS = \angle BNS$ implies AS = BS and hence S is a fixed point.

(c) Clearly the distance of the midpoint of PQ from AB is AB/4. Since it varies continuously with M, it must be the interval between the two extreme positions, so the locus is a segment length AB/2 centered over AB.

Problem B3) The planes P and Q are not parallel. The point A lies in P but not Q, and the point C lies in Q but not P. Construct points B in P and D in Q such that the quadrilateral ABCD satisfies the following conditions: (1) it lies in a plane, (2) the vertices are in the order A, B, C, D, (3) it is an isosceles trapezoid with AB parallel to CD (meaning that AD = BC, but AD is not parallel to BC unless it is a square), and (4) a circle can be inscribed in ABCD touching the sides.

بدون پاسخ