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Algebra

1 Define a sequence $< f(n) >_{n=1}^{\infty}$ of positive integers by

$$f(1) = 1$$

and

$$f(n) = \begin{cases} f(n-1) - n & \text{if } f(n-1) > n; \\ f(n-1) + n & \text{if } f(n-1) \leq n, \end{cases}$$

for $n \geq 2$. Let $S = \{n \in \mathbb{N} | f(n) = 1993\}$.

(i) Prove that S is an infinite set. **(ii)** Find the least positive integer in S. **(iii)** If all the elements of S are written in ascending order as

$$n_1 < n_2 < n_3 < \dots,$$

show that

$$\lim_{i \to \infty} \frac{n_{i+1}}{n_i} = 3.$$

2 Show that there exists a finite set $A \subset \mathbb{R}^2$ such that for every $X \in A$ there are points $Y_1, Y_2, \dots, Y_{1993}$ in A such that the distance between X and Y_i is equal to 1, for every i.

3 Prove that

$$\frac{a}{b+2c+3d} + \frac{b}{c+2d+3a} + \frac{c}{d+2a+3b} + \frac{d}{a+2b+3c} \geq \frac{2}{3}$$

for all positive real numbers a, b, c, d.

4 Solve the following system of equations, in which a is a given number satisfying $|a| > 1$:

$$\begin{aligned} x_1^2 &= ax_2 + 1 \\ x_2^2 &= ax_3 + 1 \\ &\dots \\ x_{999}^2 &= ax_{1000} + 1 \\ x_{1000}^2 &= ax_1 + 1 \end{aligned}$$

5 $a > 0$ and b, c are integers such that $ac\ \ b^2$ is a square-free positive integer P. [hide="For example"] P could be 3*5, but not $3^2 * 5$.

Let $f(n)$ be the number of pairs of integers d, e such that $ad^2 + 2bde + ce^2 = n$. Show that$f(n)$ is finite and that $f(n) = f(P^k n)$ for every positive integer k.

Original Statement:

Let a, b, c be given integers $a > 0$, $ac - b^2 = P = P_1 \cdots P_n$ where $P_1 \cdots P_n$ are (distinct) prime numbers. Let $M(n)$ denote the number of pairs of integers (x, y) for which

$$ax^2 + 2bxy + cy^2 = n.$$

Prove that $M(n)$ is finite and $M(n) = M(P_k \cdot n)$ for every integer $k \geq 0$. Note that the "n" in P_N and the "n" in $M(n)$ do not have to be the same.

$\boxed{6}$ Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Determine if there exists a strictly increasing function $f : \mathbb{N} \mapsto \mathbb{N}$ with the following properties:

(i) $f(1) = 2$;

(ii) $f(f(n)) = f(n) + n, (n \in \mathbb{N})$.

$\boxed{7}$ Let $n > 1$ be an integer and let $f(x) = x^n + 5 \cdot x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$, each having integer coefficients and degree at least one, such that $f(x) = g(x) \cdot h(x)$.

$\boxed{8}$ Let $c_1, \ldots, c_n \in \mathbb{R}$ with $n \geq 2$ such that

$$0 \leq \sum_{i=1}^{n} c_i \leq n.$$

Show that we can find integers $k_1, \ldots, k_n$ such that

$$\sum_{i=1}^{n} k_i = 0$$

and

$$1 - n \leq c_i + n \cdot k_i \leq n$$

for every $i = 1, \ldots, n$.

[hide="Another formulation:"] Let $x_1, \ldots, x_n$, with $n \geq 2$ be real numbers such that

$$|x_1 + \ldots + x_n| \leq n.$$

Show that there exist integers $k_1, \ldots, k_n$ such that

$$|k_1 + \ldots + k_n| = 0.$$

and

$$|x_i + 2 \cdot n \cdot k_i| \leq 2 \cdot n - 1$$

for every $i = 1, \ldots, n$. In order to prove this, denote $c_i = \frac{1+x_i}{2}$ for $i = 1, \ldots, n$, etc.

$\boxed{9}$ Let a, b, c, d be four non-negative numbers satisfying

$$a + b + c + d = 1.$$

Prove the inequality

$$a \cdot b \cdot c + b \cdot c \cdot d + c \cdot d \cdot a + d \cdot a \cdot b \leq \frac{1}{27} + \frac{176}{27} \cdot a \cdot b \cdot c \cdot d.$$

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Combinatorics

1 a) Show that the set $\mathbb{Q}^+$ of all positive rationals can be partitioned into three disjoint subsets. A, B, C satisfying the following conditions:

$$BA = B; B^2 = C; BC = A;$$

where HK stands for the set $\{hk : h \in H, k \in K\}$ for any two subsets H, K of $\mathbb{Q}^+$ and H^2 stands for HH.

b) Show that all positive rational cubes are in A for such a partition of $\mathbb{Q}^+$.

c) Find such a partition $\mathbb{Q}^+ = A \cup B \cup C$ with the property that for no positive integer $n \leq 34$, both n and $n+1$ are in A, that is,

$$\min\{n \in \mathbb{N} : n \in A, n+1 \in A\} > 34.$$

2 Let $n, k \in \mathbb{Z}^+$ with $k \leq n$ and let S be a set containing n distinct real numbers. Let T be a set of all real numbers of the form $x_1 + x_2 + \ldots + x_k$ where $x_1, x_2, \ldots, x_k$ are distinct elements of S. Prove that T contains at least $k(n-k)+1$ distinct elements.

3 Let $n > 1$ be an integer. In a circular arrangement of n lamps $L_0, \ldots, L_{n-1}$, each of of which can either ON or OFF, we start with the situation where all lamps are ON, and then carry out a sequence of steps, $Step_0, Step_1, \ldots$. If L_{j-1} (j is taken mod n) is ON then $Step_j$ changes the state of L_j (it goes from ON to OFF or from OFF to ON) but does not change the state of any of the other lamps. If L_{j-1} is OFF then $Step_j$ does not change anything at all. Show that:

(i) There is a positive integer $M(n)$ such that after $M(n)$ steps all lamps are ON again,

(ii) If n has the form 2^k then all the lamps are ON after $n^2 - 1$ steps,

(iii) If n has the form $2^k + 1$ then all lamps are ON after $n^2 - n + 1$ steps.

4 Let $n \geq 2, n \in \mathbb{N}$ and $A_0 = (a_{01}, a_{02}, \ldots, a_{0n})$ be any $n-$tuple of natural numbers, such that $0 \leq a_{0i} \leq i-1$, for $i = 1, \ldots, n$. $n-$tuples $A_1 = (a_{11}, a_{12}, \ldots, a_{1n}), A_2 = (a_{21}, a_{22}, \ldots, a_{2n}), \ldots$ are defined by: $a_{i+1,j} = Card\{a_{i,l} | 1 \leq l \leq j-1, a_{i,l} \geq a_{i,j}\}$, for $i \in \mathbb{N}$ and $j = 1, \ldots, n$. Prove that there exists $k \in \mathbb{N}$, such that $A_{k+2} = A_k$.

5 Let S_n be the number of sequences $(a_1, a_2, \ldots, a_n)$, where $a_i \in \{0, 1\}$, in which no six consecutive blocks are equal. Prove that $S_n \to \infty$ when $n \to \infty$.

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Geometry

1 Let ABC be a triangle, and I its incenter. Consider a circle which lies inside the circumcircle of triangle ABC and touches it, and which also touches the sides CA and BC of triangle ABC at the points D and E, respectively. Show that the point I is the midpoint of the segment DE.

2 A circle S bisects a circle S' if it cuts S' at opposite ends of a diameter. S_A, S_B,S_C are circles with distinct centers A, B, C (respectively). Show that A, B, C are collinear iff there is no unique circle S which bisects each of S_A, S_B,S_C . Show that if there is more than one circle S which bisects each of S_A, S_B,S_C , then all such circles pass through two fixed points. Find these points.

Original Statement:

A circle S is said to cut a circle Σ **diametrically** if and only if their common chord is a diameter of Σ. Let S_A, S_B, S_C be three circles with distinct centres A, B, C respectively. Prove that A, B, C are collinear if and only if there is no unique circle S which cuts each of S_A, S_B, S_C diametrically. Prove further that if there exists more than one circle S which cuts each S_A, S_B, S_C diametrically, then all such circles S pass through two fixed points. Locate these points in relation to the circles S_A, S_B, S_C.

3 Let triangle ABC be such that its circumradius is $R = 1$. Let r be the inradius of ABC and let p be the inradius of the orthic triangle $A'B'C'$ of triangle ABC. Prove that

$$p \leq 1 - \frac{1}{3 \cdot (1 + r)^2}.$$

[hide="Similar Problem posted by Pascual2005"]

Let ABC be a triangle with circumradius R and inradius r. If p is the inradius of the orthic triangle of triangle ABC, show that $\frac{p}{R} \leq 1 - \frac{\left(1+\frac{r}{R}\right)^2}{3}$.

Note. The orthic triangle of triangle ABC is defined as the triangle whose vertices are the feet of the altitudes of triangle ABC.

SOLUTION 1 by mecrazywong:

$p = 2R\cos A\cos B\cos C, 1 + \frac{r}{R} = 1 + 4\sin A/2\sin B/2\sin C/2 = \cos A + \cos B + \cos C$. Thus, the ineqaulity is equivalent to $6\cos A\cos B\cos C + (\cos A + \cos B + \cos C)^2 \leq 3$. But this is easy since $\cos A + \cos B + \cos C \leq 3/2, \cos A\cos B\cos C \leq 1/8$.

SOLUTION 2 by Virgil Nicula:

I note the inradius r' of a orthic triangle.

Must prove the inequality $\frac{r'}{R} \le 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2$.

From the wellknown relations $r' = 2R \cos A \cos B \cos C$

and $\cos A \cos B \cos C \le \frac{1}{8}$ results $\frac{r'}{R} \le \frac{1}{4}$.

But $\frac{1}{4} \le 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2 \iff \frac{1}{3}\left(1 + \frac{r}{R}\right)^2 \le \frac{3}{4} \iff$

$\left(1 + \frac{r}{R}\right)^2 \le \left(\frac{3}{2}\right)^2 \iff 1 + \frac{r}{R} \le \frac{3}{2} \iff \frac{r}{R} \le \frac{1}{2} \iff 2r \le R$ (true).

Therefore, $\frac{r'}{R} \le \frac{1}{4} \le 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2 \Longrightarrow \frac{r'}{R} \le 1 - \frac{1}{3}\left(1 + \frac{r}{R}\right)^2$.

SOLUTION 3 by darij grinberg:

I know this is not quite an ML reference, but the problem was discussed in Hyacinthos messages 6951, 6978, 6981, 6982, 6985, 6986 (particularly the last message).

4 Given a triangle ABC, let D and E be points on the side BC such that $\angle BAD = \angle CAE$. If M and N are, respectively, the points of tangency of the incircles of the triangles ABD and ACE with the line BC, then show that

$$\frac{1}{MB} + \frac{1}{MD} = \frac{1}{NC} + \frac{1}{NE}.$$

5 On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

6 For three points A, B, C in the plane, we define $m(ABC)$ to be the smallest length of the three heights of the triangle ABC, where in the case A, B, C are collinear, we set $m(ABC) = 0$. Let A, B, C be given points in the plane. Prove that for any point X in the plane,

$$m(ABC) \le m(ABX) + m(AXC) + m(XBC).$$

7 Let A, B, C, D be four points in the plane, with C and D on the same side of the line AB, such that $AC \cdot BD = AD \cdot BC$ and $\angle ADB = 90^\circ + \angle ACB$. Find the ratio

$$\frac{AB \cdot CD}{AC \cdot BD},$$

and prove that the circumcircles of the triangles ACD and BCD are orthogonal. (Intersecting circles are said to be orthogonal if at either common point their tangents are perpendicuar. Thus, proving that the circumcircles of the triangles ACD and BCD are orthogonal is equivalent to proving that the tangents to the circumcircles of the triangles ACD and BCD at the point C are perpendicular.)

8 The vertices D, E, F of an equilateral triangle lie on the sides BC, CA, AB respectively of a triangle ABC. If a, b, c are the respective lengths of these sides, and S the area of ABC, prove that

$$DE \geq \frac{2 \cdot \sqrt{2} \cdot S}{\sqrt{a^2 + b^2 + c^2 + 4 \cdot \sqrt{3} \cdot S}}.$$

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Number Theory

$\boxed{2}$ A natural number n is said to have the property P, if, for all a, n^2 divides $a^n - 1$ whenever n divides $a^n - 1$.

a.) Show that every prime number n has property P.

b.) Show that there are infinitely many composite numbers n that possess property P.

$\boxed{3}$ Let a, b, n be positive integers, $b > 1$ and $b^n - 1|a$. Show that the representation of the number a in the base b contains at least n digits different from zero.

$\boxed{4}$ Show that for any finite set S of distinct positive integers, we can find a set T S such that every member of T divides the sum of all the members of T.

Original Statement:

A finite set of (distinct) positive integers is called a **DS-set** if each of the integers divides the sum of them all. Prove that every finite set of positive integers is a subset of some **DS-set**.

$\boxed{5}$ Let S be the set of all pairs (m, n) of relatively prime positive integers m, n with n even and $m < n$. For $s = (m, n) \in S$ write $n = 2^k \cdot n_o$ where k, n_0 are positive integers with n_0 odd and define

$$f(s) = (n_0, m + n - n_0).$$

Prove that f is a function from S to S and that for each $s = (m, n) \in S$, there exists a positive integer $t \leq \frac{m+n+1}{4}$ such that

$$f^t(s) = s,$$

where

$$f^t(s) = \underbrace{(f \circ f \circ \cdots \circ f)}_{t \text{ times}}(s).$$

If $m + n$ is a prime number which does not divide $2^k - 1$ for $k = 1, 2, \ldots, m + n - 2$, prove that the smallest value t which satisfies the above conditions is $\left[\frac{m+n+1}{4}\right]$ where $[x]$ denotes the greatest integer $\leq x$.