

# IMO Shortlist 1994

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## Algebra

- [1] Let  $a_0 = 1994$  and  $a_{n+1} = \frac{a_n^2}{a_n+1}$  for each nonnegative integer  $n$ . Prove that  $1994 - n$  is the greatest integer less than or equal to  $a_n$ ,  $0 \leq n \leq 998$
- [2] Let  $m$  and  $n$  be two positive integers. Let  $a_1, a_2, \dots, a_m$  be  $m$  different numbers from the set  $\{1, 2, \dots, n\}$  such that for any two indices  $i$  and  $j$  with  $1 \leq i < j \leq m$  and  $a_i + a_j \leq n$ , there exists an index  $k$  such that  $a_i + a_j = a_k$ . Show that

$$\frac{a_1 + a_2 + \dots + a_m}{m} \geq \frac{n+1}{2}.$$

- [3] Let  $S$  be the set of all real numbers strictly greater than 1. Find all functions  $f : S \rightarrow S$  satisfying the two conditions:
- (a)  $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$  for all  $x, y$  in  $S$ ;
- (b)  $\frac{f(x)}{x}$  is strictly increasing on each of the two intervals  $-1 < x < 0$  and  $0 < x$ .
- [4] Let  $\mathbb{R}$  denote the set of all real numbers and  $\mathbb{R}^+$  the subset of all positive ones. Let  $\alpha$  and  $\beta$  be given elements in  $\mathbb{R}$ , not necessarily distinct. Find all functions  $f : \mathbb{R}^+ \mapsto \mathbb{R}$  such that

$$f(x)f(y) = y^\alpha f\left(\frac{x}{2}\right) + x^\beta f\left(\frac{y}{2}\right) \forall x, y \in \mathbb{R}^+.$$

- [5] Let  $f(x) = \frac{x^2+1}{2x}$  for  $x \neq 0$ . Define  $f^{(0)}(x) = x$  and  $f^{(n)}(x) = f(f^{(n-1)}(x))$  for all positive integers  $n$  and  $x \neq 0$ . Prove that for all non-negative integers  $n$  and  $x \neq \{-1, 0, 1\}$

$$\frac{f^{(n)}(x)}{f^{(n+1)}(x)} = 1 + \frac{1}{f\left(\left(\frac{x+1}{x-1}\right)^{2n}\right)}.$$

## Combinatorics

- 1 Two players play alternately on a  $5 \times 5$  board. The first player always enters a 1 into an empty square and the second player always enters a 0 into an empty square. When the board is full, the sum of the numbers in each of the nine  $3 \times 3$  squares is calculated and the first player's score is the largest such sum. What is the largest score the first player can make, regardless of the responses of the second player?
- 2 In a certain city, age is reckoned in terms of real numbers rather than integers. Every two citizens  $x$  and  $x'$  either know each other or do not know each other. Moreover, if they do not, then there exists a chain of citizens  $x = x_0, x_1, \dots, x_n = x'$  for some integer  $n \geq 2$  such that  $x_{i-1}$  and  $x_i$  know each other. In a census, all male citizens declare their ages, and there is at least one male citizen. Each female citizen provides only the information that her age is the average of the ages of all the citizens she knows. Prove that this is enough to determine uniquely the ages of all the female citizens.
- 3 Peter has three accounts in a bank, each with an integral number of dollars. He is only allowed to transfer money from one account to another so that the amount of money in the latter is doubled. Prove that Peter can always transfer all his money into two accounts. Can Peter always transfer all his money into one account?
- 4 There are  $n + 1$  cells in a row labeled from 0 to  $n$  and  $n + 1$  cards labeled from 0 to  $n$ . The cards are arbitrarily placed in the cells, one per cell. The objective is to get card  $i$  into cell  $i$  for each  $i$ . The allowed move is to find the smallest  $h$  such that cell  $h$  has a card with a label  $k > h$ , pick up that card, slide the cards in cells  $h + 1, h + 2, \dots, k$  one cell to the left and to place card  $k$  in cell  $k$ . Show that at most  $2^n - 1$  moves are required to get every card into the correct cell and that there is a unique starting position which requires  $2^n - 1$  moves. [For example, if  $n = 2$  and the initial position is 210, then we get 102, then 012, a total of 2 moves.]
- 5 1994 girls are seated at a round table. Initially one girl holds  $n$  tokens. Each turn a girl who is holding more than one token passes one token to each of her neighbours.
  - a.) Show that if  $n < 1994$ , the game must terminate.
  - b.) Show that if  $n = 1994$  it cannot terminate.
- 6 Two players play alternatively on an infinite square grid. The first player puts a X in an empty cell and the second player puts a O in an empty cell. The first player wins if he gets 11 adjacent Xs in a line, horizontally, vertically or diagonally. Show that the second player can always prevent the first player from winning.

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- 7 Let  $n > 2$ . Show that there is a set of  $2^n - 1$  points in the plane, no three collinear such that no  $2n$  form a convex  $2n$ -gon.

## Geometry

- 1  $C$  and  $D$  are points on a semicircle. The tangent at  $C$  meets the extended diameter of the semicircle at  $B$ , and the tangent at  $D$  meets it at  $A$ , so that  $A$  and  $B$  are on opposite sides of the center. The lines  $AC$  and  $BD$  meet at  $E$ .  $F$  is the foot of the perpendicular from  $E$  to  $AB$ . Show that  $EF$  bisects angle  $CFD$
- 2  $ABCD$  is a quadrilateral with  $BC$  parallel to  $AD$ .  $M$  is the midpoint of  $CD$ ,  $P$  is the midpoint of  $MA$  and  $Q$  is the midpoint of  $MB$ . The lines  $DP$  and  $CQ$  meet at  $N$ . Prove that  $N$  is inside the quadrilateral  $ABCD$ .
- 3 A circle  $C$  has two parallel tangents  $L'$  and  $L''$ . A circle  $C'$  touches  $L'$  at  $A$  and  $C$  at  $X$ . A circle  $C''$  touches  $L''$  at  $B$ ,  $C$  at  $Y$  and  $C'$  at  $Z$ . The lines  $AY$  and  $BX$  meet at  $Q$ . Show that  $Q$  is the circumcenter of  $XYZ$
- 4 Let  $ABC$  be an isosceles triangle with  $AB = AC$ .  $M$  is the midpoint of  $BC$  and  $O$  is the point on the line  $AM$  such that  $OB$  is perpendicular to  $AB$ .  $Q$  is an arbitrary point on  $BC$  different from  $B$  and  $C$ .  $E$  lies on the line  $AB$  and  $F$  lies on the line  $AC$  such that  $E, Q, F$  are distinct and collinear. Prove that  $OQ$  is perpendicular to  $EF$  if and only if  $QE = QF$ .
- 5 A circle  $C$  with center  $O$ . and a line  $L$  which does not touch circle  $C$ .  $OQ$  is perpendicular to  $L$ ,  $Q$  is on  $L$ .  $P$  is on  $L$ , draw two tangents  $L_1, L_2$  to circle  $C$ .  $QA, QB$  are perpendicular to  $L_1, L_2$  respectively. ( $A$  on  $L_1$ ,  $B$  on  $L_2$ ). Prove that, line  $AB$  intersect  $QO$  at a fixed point.

*Original formulation:*

A line  $l$  does not meet a circle  $\omega$  with center  $O$ .  $E$  is the point on  $l$  such that  $OE$  is perpendicular to  $l$ .  $M$  is any point on  $l$  other than  $E$ . The tangents from  $M$  to  $\omega$  touch it at  $A$  and  $B$ .  $C$  is the point on  $MA$  such that  $EC$  is perpendicular to  $MA$ .  $D$  is the point on  $MB$  such that  $ED$  is perpendicular to  $MB$ . The line  $CD$  cuts  $OE$  at  $F$ . Prove that the location of  $F$  is independent of that of  $M$ .

## Number Theory

- 1  $M$  is a subset of  $\{1, 2, 3, \dots, 15\}$  such that the product of any three distinct elements of  $M$  is not a square. Determine the maximum number of elements in  $M$ .
- 2 Find all ordered pairs  $(m, n)$  where  $m$  and  $n$  are positive integers such that  $\frac{n^3+1}{mn-1}$  is an integer.
- 3 Show that there exists a set  $A$  of positive integers with the following property: for any infinite set  $S$  of primes, there exist *two* positive integers  $m$  in  $A$  and  $n$  not in  $A$ , each of which is a product of  $k$  distinct elements of  $S$  for some  $k \geq 2$ .
- 4 Define the sequences  $a_n, b_n, c_n$  as follows.  $a_0 = k, b_0 = 4, c_0 = 1$ . If  $a_n$  is even then  $a_{n+1} = \frac{a_n}{2}$ ,  $b_{n+1} = 2b_n, c_{n+1} = c_n$ . If  $a_n$  is odd, then  $a_{n+1} = a_n - \frac{b_n}{2} - c_n, b_{n+1} = b_n, c_{n+1} = b_n + c_n$ . Find the number of positive integers  $k < 1995$  such that some  $a_n = 0$ .
- 5 For any positive integer  $k$ , let  $f_k$  be the number of elements in the set  $\{k+1, k+2, \dots, 2k\}$  whose base 2 representation contains exactly three 1s.
  - (a) Prove that for any positive integer  $m$ , there exists at least one positive integer  $k$  such that  $f(k) = m$ .
  - (b) Determine all positive integers  $m$  for which there exists *exactly one*  $k$  with  $f(k) = m$ .
- 6 Define the sequence  $a_1, a_2, a_3, \dots$  as follows.  $a_1$  and  $a_2$  are coprime positive integers and  $a_{n+2} = a_{n+1}a_n + 1$ . Show that for every  $m > 1$  there is an  $n > m$  such that  $a_m^m$  divides  $a_n^n$ . Is it true that  $a_1$  must divide  $a_n^n$  for some  $n > 1$ ?
- 7 A wobbly number is a positive integer whose digits are alternately zero and non-zero with the last digit non-zero (for example, 201). Find all positive integers which do not divide any wobbly number.