IMO Shortlist 2003

Algebra

1 Let a_{ij} (with the indices i and j from the set $\{1,\ 2,\ 3\}$) be real numbers such that

$a_{ij} > 0$ for $i = j$; $a_{ij} < 0$ for $i \neq j$.

Prove the existence of positive real numbers c_1, c_2, c_3 such that the numbers

$a_{11}c_1 + a_{12}c_2 + a_{13}c_3$, $a_{21}c_1 + a_{22}c_2 + a_{23}c_3$, $a_{31}c_1 + a_{32}c_2 + a_{33}c_3$

are either all negative, or all zero, or all positive.

2 Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that (i) $f(0) = 0, f(1) = 1$; (ii) $f(a)+f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a, b such that $a < 1 < b$.

3 Consider two monotonically decreasing sequences (a_k) and (b_k), where $k \geq 1$, and a_k and b_k are positive real numbers for every k. Now, define the sequences

$c_k = \min{(a_k, b_k)}$; $A_k = a_1 + a_2 + ... + a_k$; $B_k = b_1 + b_2 + ... + b_k$; $C_k = c_1 + c_2 + ... + c_k$

for all natural numbers k.

(a) Do there exist two monotonically decreasing sequences (a_k) and (b_k) of positive real numbers such that the sequences (A_k) and (B_k) are not bounded, while the sequence (C_k) is bounded?

(b) Does the answer to problem **(a)** change if we stipulate that the sequence (b_k) must be $b_k = \frac{1}{k}$ for all k ?

4 Let n be a positive integer and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be real numbers. Prove that

$$\left(\sum_{i,j=1}^{n} |x_i - x_j|\right)^2 \leq \frac{2(n^2 - 1)}{3} \sum_{i,j=1}^{n} (x_i - x_j)^2.$$

Show that the equality holds if and only if $x_1, \dots, x_n$ is an arithmetic sequence.

5 Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ that satisfy the following conditions:

- $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;

- $f(x) < f(y)$ for all $1 \leq x < y$.

6 Let n be a positive integer and let $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ be two sequences of positive real numbers. Suppose $(z_2, \dots, z_{2n})$ is a sequence of positive real numbers such that $z_{i+j}^2 \geq x_i y_j$ for all $1 \leq i, j \leq n$.

Let $M = \max\{z_2, \ldots, z_{2n}\}$. Prove that

$$\left(\frac{M + z_2 + \cdots + z_{2n}}{2n}\right)^2 \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

[hide="comment"] *Edited by Orl.*

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Combinatorics

$\boxed{1}$ Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers t_1, $t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\}, \qquad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

$\boxed{2}$ Let D_1, D_2, ..., D_n be closed discs in the plane. (A closed disc is the region limited by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1 = 14020$ other discs D_i.

$\boxed{3}$ Let $n \geq 5$ be an integer. Find the maximal integer k such that there exists a polygon with n vertices (convex or not, but not self-intersecting!) having k internal 90° angles.

$\boxed{4}$ Given n real numbers x_1, x_2, ..., x_n, and n further real numbers y_1, y_2, ..., y_n. The elements a_{ij} (with $1 \leq i,\ j \leq n$) of an $n \times n$ matrix are defined as follows:

$a_{ij} = \begin{cases} 1 & \text{if} \quad x_i + y_j \geq 0; \\ 0 & \text{if} \quad x_i + y_j < 0. \end{cases}$

Further, let B be an $n \times n$ matrix whose elements are numbers from the set 0; 1 satisfying the following condition: The sum of all elements of each row of B equals the sum of all elements of the corresponding row of A; the sum of all elements of each column of B equals the sum of all elements of the corresponding column of A. Show that in this case, A = B.

[hide="comment"] (This one is from the ISL 2003, but in any case, [url=http://www.bundeswettbewerb-mathematik.de/imo/aufgaben/aufgaben.htm]the official problems and solutions - in German -[/url] are already online, hence I take the liberty to post it here.)

Darij

$\boxed{5}$ Regard a plane with a Cartesian coordinate system; for each point with integer coordinates, draw a circular disk centered at this point and having the radius $\frac{1}{1000}$.

a) Prove the existence of an equilateral triangle whose vertices lie in the interior of different disks;

b) Show that every equilateral triangle whose vertices lie in the interior of different disks has a sidelength ¿ 96.

Radu Gologan, Romania [hide="Remark"] [The "¿ 96" in **(b)** can be strengthened to "¿ 124". By the way, part **(a)** of this problem is the place where I used [url=http://mathlinks.ro/viewtopic.php?t=5537 well-known "Dedekind" theorem[/url].]

$\boxed{6}$ Let $f(k)$ be the number of all non-negative integers n satisfying the following conditions:

(1) The integer n has exactly k digits in the decimal representation (where the first digit is not necessarily non-zero!), i. e. we have $0 \leq n < 10^k$.

(2) These k digits of n can be permuted in such a way that the resulting number is divisible by 11.

Show that for any positive integer number m, we have $f(2m) = 10f(2m-1)$.

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Geometry

1. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

2. Given three fixed pairwisely distinct points A, B, C lying on one straight line in this order. Let G be a circle passing through A and C whose center does not lie on the line AC. The tangents to G at A and C intersect each other at a point P. The segment PB meets the circle G at Q.

   Show that the point of intersection of the angle bisector of the angle AQC with the line AC does not depend on the choice of the circle G.

3. Let ABC be a triangle, and P a point in the interior of this triangle. Let D, E, F be the feet of the perpendiculars from the point P to the lines BC, CA, AB, respectively. Assume that

   $AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2$.

   Furthermore, let I_a, I_b, I_c be the excenters of triangle ABC. Show that the point P is the circumcenter of triangle $I_aI_bI_c$.

4. Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

5. Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

   [hide="comment"] (According to my team leader, last year some of the countries wanted a geometry question that was even easier than this...that explains IMO 2003/4...)

   [Note by Darij: This was also Problem 6 of the German pre-TST 2004, written in December 03.]

   Edited by Orl.

$\boxed{6}$ Each pair of opposite sides of a convex hexagon has the following property: the distance between their midpoints is equal to $\dfrac{\sqrt{3}}{2}$ times the sum of their lengths. Prove that all the angles of the hexagon are equal.

$\boxed{7}$ Let ABC be a triangle with semiperimeter s and inradius r. The semicircles with diameters BC, CA, AB are drawn on the outside of the triangle ABC. The circle tangent to all of these three semicircles has radius t. Prove that

$$\frac{s}{2} < t \le \frac{s}{2} + \left(1 - \frac{\sqrt{3}}{2}\right) r.$$

Alternative formulation. In a triangle ABC, construct circles with diameters BC, CA, and AB, respectively. Construct a circle w externally tangent to these three circles. Let the radius of this circle w be t. Prove: $\frac{s}{2} < t \le \frac{s}{2} + \frac{1}{2}\left(2 - \sqrt{3}\right) r$, where r is the inradius and s is the semiperimeter of triangle ABC.

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Number Theory

1. Let m be a fixed integer greater than 1. The sequence x_0, x_1, x_2, $\ldots$ is defined as follows: $x_i = 2^i$ if $0 \le i \le m-1$ and $x_i = \sum_{j=1}^{m} x_{i-j}$, if $i \ge m$.

2. Each positive integer a is subjected to the following procedure, yielding the number $d = d(a)$:

   (a) The last digit of a is moved to the first position. The resulting number is called b. **(b)** The number b is squared. The resulting number is called c. **(c)** The first digit of c is moved to the last position. The resulting number is called d.

   (All numbers are considered in the decimal system.) For instance, $a = 2003$ gives $b = 3200$, $c = 10240000$ and $d = 02400001 = 2400001 = d(2003)$.

   Find all integers a such that $d(a) = a^2$.

3. Determine all pairs of positive integers (a, b) such that

   $$\frac{a^2}{2ab^2 - b^3 + 1}$$

   is a positive integer.

4. Let b be an integer greater than 5. For each positive integer n, consider the number

   $$x_n = \underbrace{11\cdots1}_{n-1}\underbrace{22\cdots2}_{n}5,$$

   written in base b.

   Prove that the following condition holds if and only if $b = 10$:

   there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

5. An integer n is said to be *good* if $|n|$ is not the square of an integer. Determine all integers m with the following property: m can be represented, in infinitely many ways, as a sum of three distinct good integers whose product is the square of an odd integer.

6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n, the number $n^p - p$ is not divisible by q.

7. The sequence a_0, a_1, a_2, $\ldots$ is defined as follows: $a_0 = 2$, $\quad a_{k+1} = 2a_k^2 - 1 \quad$ for $k \ge 0$. Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

[hide=”comment”] Hi guys ,

Here is a nice problem:

Let be given a sequence a_n such that $a_0 = 2$ and $a_{n+1} = 2a_n^2 - 1$. Show that if p is an odd prime such that $p|a_n$ then we have $p^2 \equiv 1 \pmod{2^{n+3}}$

Here are some futher question proposed by me ¡!– s:P –¿¡img src=”SMILIES$_PATH/tongue.gif”alt =$ ” $: P”title = ”Razz”/ ><!--s : P -- > roveordisprovethat$: 1)gcd(n,a$_n$) = 1 2) for every odd prime number p we have $a_m \equiv \pm 1 \pmod{p}$ where $m = \frac{p^2-1}{2^k}$ where $k = 1$ or 2

Thanks kiu si u

Edited by Orl.

8 Let p be a prime number and let A be a set of positive integers that satisfies the following conditions: **(1)** the set of prime divisors of the elements in A consists of $p - 1$ elements; **(2)** for any nonempty subset of A, the product of its elements is not a perfect p-th power. What is the largest possible number of elements in A ?