IMO Shortlist 2004

Algebra

1 Let $n \geq 3$ be an integer. Let t_1, t_2, ..., t_n be positive real numbers such that

$n^2 + 1 > (t_1 + t_2 + ... + t_n) \left(\frac{1}{t_1} + \frac{1}{t_2} + ... + \frac{1}{t_n} \right)$.

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

2 Let a_0, a_1, a_2, ... be an infinite sequence of real numbers satisfying the equation $a_n = |a_{n+1} - a_{n+2}|$ for all $n \geq 0$, where a_0 and a_1 are two different positive reals.

Can this sequence a_0, a_1, a_2, ... be bounded?

[hide="Remark"] This one is from the IMO Shortlist 2004, but it's already published on the [url=http://www.bundeswettbewerb-mathematik.de/imo/aufgaben/aufgaben.htm]official BWM website[/url] und thus I take the freedom to post it here:

3 Does there exist a function $s\colon \mathbf{Q} \to \{-1, 1\}$ such that if x and y are distinct rational numbers satisfying $xy = 1$ or $x + y \in \{0, 1\}$, then $s(x)s(y) = -1$? Justify your answer.

[hide="comment"] *Edited by orl.*

4 Find all polynomials f with real coefficients such that for all reals a, b, c such that $ab+bc+ca = 0$ we have the following relations

$$f(a - b) + f(b - c) + f(c - a) = 2f(a + b + c).$$

5 If a, b ,c are three positive real numbers such that $ab + bc + ca = 1$, prove that

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \leq \frac{1}{abc}.$$

6 Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$f\left(x^2 + y^2 + 2f\left(xy\right)\right) = \left(f\left(x + y\right)\right)^2$

for all $x, y \in \mathbb{R}$.

[hide="comment"] *Edited by orl.*

7 Let $a_1, a_2, \ldots, a_n$ be positive real numbers, $n > 1$. Denote by g_n their geometric mean, and by $A_1, A_2, \ldots, A_n$ the sequence of arithmetic means defined by

$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}, \qquad k = 1, 2, \ldots, n.$

Let G_n be the geometric mean of $A_1, A_2, \ldots, A_n$. Prove the inequality

$n\sqrt[n]{\frac{G_n}{A_n}}+\frac{g_n}{G_n}\le n+1$

and establish the cases of equality.

IMO Shortlist 2004

Combinatorics

1. There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

   i.) Each pair of students are in exactly one club.

   ii.) For each student and each society, the student is in exactly one club of the society.

   iii.) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is in exactly m societies.

   Find all possible values of k. [hide="Remark"]In IMOTC 2005, it is given that $m = 2005$.

2. Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwisely distinct (i. e. no three circles have a common point). No three circles have a point in common. Each intersection point must be colored with one of n distinct colors so that each color is used at least once and exactly k distinct colors occur on each circle. Find all values of $n \geq 2$ and k for which such a coloring is possible.

   [hide="comment"] This one is from the IMO Shortlist 2004, but it's already published on the [url=http://www.bundeswettbewerb-mathematik.de/imo/aufgaben/aufgaben.htm]official BWM website[/url] und thus I take the freedom to post it here.

   Edited by orl.

3. In a country there are $n > 3$ cities. We can add a street between 2 cities A and B, iff there exist 2 cities X and Y different from A and B such that there is no street between A and X, X and Y, and Y and B. Find the biggest number of streets one can construct!

   [hide="Official Wording"] The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices (where each pair of vertices are joined by an edge).

4. Consider a matrix of size $n \times n$ whose entries are real numbers of absolute value not exceeding 1. The sum of all entries of the matrix is 0. Let n be an even positive integer. Determine the least number C such that every such matrix necessarily has a row or a column with the sum of its entries not exceeding C in absolute value.

$\boxed{5}$ 3. A and B play a game, given an integer N, A writes down 1 first, then every player sees the last number written and if it is n then in his turn he writes $n+1$ or $2n$, but his number cannot be bigger than N. The player who writes N wins!, For wich values of N does B wins?

$\boxed{6}$ For an $n \times n$ matrix A, let X_i be the set of entries in row i, and Y_j the set of entries in column j, $1 \leq i, j \leq n$. We say that A is *golden* if $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ are distinct sets. Find the least integer n such that there exists a 2004×2004 golden matrix with entries in the set $\{1, 2, \ldots, n\}$.

[hide="comment"] **That's problem 3 of the 2nd German TST 2005 and Moldova TST 2005.** *Edited by orl.*

$\boxed{7}$ Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure. [img]http://www.mathlinks.ro/Forum/files/hook.png[/img] Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that

- the rectangle is covered without gaps and without overlaps - no part of a hook covers area outside the rectagle.

$\boxed{8}$ For a finite graph G, let $f(G)$ be the number of triangles and $g(G)$ the number of tetrahedra formed by edges of G. Find the least constant c such that

$g(G)^3 \leq c \cdot f(G)^4$

for every graph G.

Geometry

1 1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

2 Let Γ be a circle and let d be a line such that Γ and d have no common points. Further, let AB be a diameter of the circle Γ; assume that this diameter AB is perpendicular to the line d, and the point B is nearer to the line d than the point A. Let C be an arbitrary point on the circle Γ, different from the points A and B. Let D be the point of intersection of the lines AC and d. One of the two tangents from the point D to the circle Γ touches this circle Γ at a point E; hereby, we assume that the points B and E lie in the same halfplane with respect to the line AC. Denote by F the point of intersection of the lines BE and d. Let the line AF intersect the circle Γ at a point G, different from A.

Prove that the reflection of the point G in the line AB lies on the line CF.

3 Let ABC be an acute-angled triangle such that $\angle ABC < \angle ACB$, let O be the circumcenter of triangle ABC, and let $D = AO \cap BC$. Denote by E and F the circumcenters of triangles ABD and ACD, respectively. Let G be a point on the extension of the segment AB beyound A such that $AG = AC$, and let H be a point on the extension of the segment AC beyound A such that $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

[hide="comment"] [hide="Official version"] Let O be the circumcenter of an acute-angled triangle ABC with $\angle B < \angle C$. The line AO meets the side BC at D. The circumcenters of the triangles ABD and ACD are E and F, respectively. Extend the sides BA and CA beyond A, and choose on the respective extensions points G and H such that $AG = AC$ and $AH = AB$. Prove that the quadrilateral $EFGH$ is a rectangle if and only if $\angle ACB - \angle ABC = 60^\circ$.

Edited by orl.

4 In a convex quadrilateral $ABCD$, the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$\angle PBC = \angle DBA$ and $\angle PDC = \angle BDA$.

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

5 Let $A_1A_2A_3...A_n$ be a regular n-gon. Let B_1 and B_n be the midpoints of its sides A_1A_2 and $A_{n-1}A_n$. Also, for every $i \in \{2; 3; 4; ...; n-1\}$, let S be the point of intersection of the

lines A_1A_{i+1} and A_nA_i, and let B_i be the point of intersection of the angle bisector bisector of the angle $\measuredangle A_iSA_{i+1}$ with the segment A_iA_{i+1}.

Prove that: $\sum_{i=1}^{n-1} \measuredangle A_1B_iA_n = 180^\circ$.

$\boxed{6}$ Let P be a convex polygon. Prove that there exists a convex hexagon that is contained in P and whose area is at least $\frac{3}{4}$ of the area of the polygon P.

Alternative version. Let P be a convex polygon with $n \geq 6$ vertices. Prove that there exists a convex hexagon with

a) vertices on the sides of the polygon (or) **b)** vertices among the vertices of the polygon

such that the area of the hexagon is at least $\frac{3}{4}$ of the area of the polygon.

I couldn't solve this one, partially because I'm not quite sure of the statements' meaning (a) or (b) ¡!– s:) –¿¡img src="SMILIES$_P$*ATH/smile.gif*" *alt* = " :)"*title* = "*Smile*"/ ><! − −*s* :) − − > *Obviouslyif* **a)***istruethensois***b)**.*Foragiventriangle* ABC, *let* X*beavariablepointontheline* BC*suchthat* C*liesbetween* B*and* X*andtheincirclesofthetriangles* ABX*and* ACX*intersectattwodistinctpoint* P*and* Q.*Provethattheline* PQ*passesthroughapointindependentof* X.

[hide="comment"] An extension by [url=http://www.mathlinks.ro/Forum/profile.php?mode=viewprofileu= Grinberg[/url] can be found [url=http://www.mathlinks.ro/Forum/viewtopic.php?t=40772]here.[/url]

$\boxed{8}$ Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.

IMO Shortlist 2004

Number Theory

$\boxed{1}$ Let $\tau(n)$ denote the number of positive divisors of the positive integer n. Prove that there exist infinitely many positive integers a such that the equation $\tau(an) = n$ does not have a positive integer solution n.

$\boxed{2}$ The function f from the set $\mathbf{N}$ of positive integers into itself is defined by the equality

$f(n) = \sum_{k=1}^{n} \gcd(k, n), \qquad n \in \mathbf{N},$

a) Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbf{N}$.

b) Prove that for each $a \in \mathbf{N}$ the equation $f(x) = ax$ has a solution.

c) Find all $a \in \mathbf{N}$ such that the equation $f(x) = ax$ has a unique solution.

[hide="comment"] *Edited by orl.*

$\boxed{3}$ Find all functions $f : \mathbb{N}^* \to \mathbb{N}^*$ satisfying

$$\left(f^2\left(m\right) + f\left(n\right)\right) \mid \left(m^2 + n\right)^2$$

for any two positive integers m and n.

Remark. The abbreviation $\mathbb{N}^*$ stands for the set of all positive integers: $\mathbb{N}^* = \{1, 2, 3, ...\}$. By $f^2\left(m\right)$, we mean $\left(f\left(m\right)\right)^2$ (and not $f\left(f\left(m\right)\right)$).

$\boxed{4}$ Let k be a fixed integer greater than 1, and let $m = 4k^2 - 5$. Show that there exist positive integers a and b such that the sequence (x_n) defined by

$x_0 = a, \quad x_1 = b, \quad x_{n+2} = x_{n+1} + x_n \quad \text{for} \quad n = 0, 1, 2, \dots,$

has all of its terms relatively prime to m.

$\boxed{5}$ We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers n such that n has a multiple which is alternating.

$\boxed{6}$ Given an integer $n > 1$, denote by P_n the product of all positive integers x less than n and such that n divides $x^2 - 1$. For each $n > 1$, find the remainder of P_n on division by n.

$\boxed{7}$ Let p be an odd prime and n a positive integer. In the coordinate plane, eight distinct points with integer coordinates lie on a circle with diameter of length p^n. Prove that there exists a triangle with vertices at three of the given points such that the squares of its side lengths are integers divisible by p^{n+1}.